PHYSICAL REVIEW LETTERS

VOLUME 48 | 14 JUNE 1982 | NUMBER 24

Extrema of Landau and Higgs Polynomials and Fixed Points of Renormalization-Group Equations

Marko V. Jarić
Institut des Hautes Etudes Scientifiques, F-91440 Bures-sur-Yvette, France
(Received 29 March 1982)

A powerful method is presented for solving systems of nonlinear equations such as those occurring in the Landau theory of phase transitions, in the Higgs mechanism of spontaneous symmetry breaking, and in renormalization-group studies of critical phenomena. As an illustration a ferroelectric phase transition in perovskites is considered with the most general free energy of sixth degree. A special case is a fourth-degree potential which corresponds, e.g., to an SO(7), adjoint representation, Higgs problem.

PACS numbers: 05.70.Fk, 11.15.Ex, 12.10.-g, 64.60.-i

Systems of nonlinear (polynomial) equations occur often in physics. For example, such equations occur in the Landau theory of phase transitions and in the Higgs mechanism of spontaneous symmetry breaking. The solutions determine low, broken, symmetries. In a renormalization-group approach to critical phenomena these equations are fixed-point equations. The solutions and their stability determine critical behavior of a physical system. Many papers have been devoted to a general study of these equations.[1-4]

The aforementioned equations,

$$\underline{f}(\underline{\psi})=0, \tag{1}$$

transform as a vector under the action of a group R. That is, for every transformation r in the group R we have

$$\underline{f}(r\underline{\psi})=r\underline{f}(\underline{\psi}). \tag{2}$$

R is an orthogonal representation (a group of $n\times n$ orthogonal matrices) of a physical group. R acts on a real vector space $\mathcal{R}^n$ spanned by the variables $\underline{\psi}$. In the Landau theory R represents a symmetry group of the high-symmetry phase. In the Higgs mechanism it represents a gauge group, whereas in a renormalization-group approach R is a group which commutes with the renormalization-group transformations.

It was suggested in a work on symmetry properties of renormalization-group equations[3] that a search for the solutions should be conducted by examining in a systematic fashion particular invariant subspaces of the parameter space. The basic idea is simple: Since $\underline{f}(\underline{\psi})$ is a vector field it cannot cross any symmetry hyperplane in $\mathcal{R}^n$ [an i-dimensional ($0\leq i\leq n$) linear subspace of $\mathcal{R}^n$ we call a hyperplane; a symmetry hyperplane is fixed under the action of a group $L\subseteq R$ which acts nontrivially on vectors perpendicular to the hyperplane; a rotation axis ($i=1$) or a reflection plane ($i=2$) is a symmetry hyperplane]. At such a plane the components of $\underline{f}(\underline{\psi})$ perpendicular to the hyperplane are identically zero. Consequently, solutions of Eq. (1) need to be found only within such hyperplanes of lower and lower symmetry. An underlying physical idea is that the symmetry of a physical problem which corresponds to a solution is expected to be minimally broken.

Without loss of generality we will focus our attention on the Landau theory of ferroelectrics in which case R is finite (in the case of compact groups used in the Higgs mechanism the group relevant to the symmetry breaking is often a finite point group[5]). For some of the group-theoretical jargon the reader is referred to Ref. 6.

In Landau theory the free energy $F(\underline{\psi})$ is an R-invariant polynomial of degree $q+1 \geq 4$. The minima of $F(\underline{\psi})$ are determined by Eq. (1) with $\underline{f}(\underline{\psi}) \equiv \partial F(\underline{\psi})$. R invariance of F insures the validity of Eq. (2). For the sake of simplicity we will also assume R to be irreducible.

In order to determine symmetry hyperplanes in $\mathcal{R}^n$ it is simplest to use little (isotropy) groups of R. A little group L defines an $i(L)$-dimensional symmetry hyperplane fix$L$ as the subspace of $\mathcal{R}^n$ of all vectors left invariant by $L \subseteq R$ (L is the centralizer of fix$L$). The problem of finding all symmetry hyperplanes reduces to the problem of finding all little groups L. The latter has been solved in the general case of countable groups by the chain criterion.[7] A straightforward method requires only knowledge of the characters and subgroups of R. For infinite groups the chain criterion is a restrictive necessary condition.

The second step is to find a projector $\mathcal{P}(L)$ which projects an arbitrary vector $\underline{\psi} \in \mathcal{R}^n$ onto a hyperplane fix$L$. $\mathcal{P}(L)$ may be constructed by using standard techniques. Using this projector we may write a linear equation of the hyperplane fix$L$ by

$$[1-\mathcal{P}(L)]\underline{\psi}=0. \tag{3}$$

Since at a point satisfying Eq. (3) $\underline{f}(\underline{\psi})$ is tangential to the hyperplane fix$L$, it follows that Eq. (1) may be replaced by

$$\mathcal{P}(L)\underline{f}(\underline{\psi})=0. \tag{4}$$

Equation (3) gives $n-i(L)$ independent linear equations for $\underline{\psi}$ while Eq. (4) contains the remaining $i(L)$ independent equations.

If we observe that fix$L_s \subset$ fix$L$ holds for a little group L_s, $L_s \supset L$, it becomes apparent that Eq. (4) will contain the solutions from fixL_s. Let us assume that there are m little groups L_s, $s=1,\ldots,m$, which satisfy $L_s \supset L$ and $i(L_s)=i(L)+1$. The equation for vectors of fix$L$ which are in fixL_s is then

$$[\mathcal{P}(L)-\mathcal{P}(L_s)]\underline{\psi}=0. \tag{5}$$

Since at such a point $\underline{\psi}$ a component of $\underline{f}(\underline{\psi})$ perpendicular to fixL_s (in fix$L$) must be identically zero it follows that it will contain Eq. (5) as a factor:

$$[\mathcal{P}(L)-\mathcal{P}(L_s)]\underline{f}(\underline{\psi}_L) \equiv g_s(\underline{\psi}_L)[\mathcal{P}(L)-\mathcal{P}(L_s)]\underline{\psi}_L=0, \tag{6}$$

where I have used an extra subscript L to emphasize that Eq. (6) holds for $\underline{\psi}$ from fix$L$, i.e., for $\underline{\psi}$ satisfying Eq. (3). I have also defined (on fix$L$) a new function $g_s(\underline{\psi}_L)$ which is of degree $q-1$. Furthermore, typically $m>i(L)$ and it is possible to find $i(L)$ hyperplanes fixL_s whose normals span the whole space fix$L$. Let us suppose that these are numbered $s=1,\ldots,i(L)$. In such a case we can replace Eq. (5) by $i(L)$ equations (6) which are independent for $s=1,\ldots,i(L)$:

$$g_s(\underline{\psi}_L)=0, \quad s=1,\ldots,i(L), \tag{7}$$

where I have removed a factor of Eq. (5) as these solutions are from fixL_s.

It is important that in going from Eq. (4) to Eq. (7) we succeeded in reducing the degree of the equations by one. However, it is clear that Eqs. (7) should also contain solutions Eq. (5) for $s=i(L)+1,\ldots,m$. Thus certain linear combinations of Eqs. (7) are expected to have these solutions as factors, which would further reduce the degree of the relevant equations. It is remarkable that a search for these factorizations may also be systematized. Namely, we observe that the *hyperplane* fix$L$ is in fact invariant under a group $N(L)$, $L \subseteq N(L) \subseteq R$, which is a normalizer of L in R. That is, $N(L)$ is the largest subgroup of R which contains L as a normal subgroup. $N(L)$ is also a normalizer of fix$L$. When it satisfies $N(L) \neq L$, it acts nontrivially on fix$L$ [in fact, the quotient group $N(L)/L$ acts effectively on fix$L$]. Thus there is a group action defined on vectors $\underline{\psi}_L \in$ fix$L$. $g_s(\underline{\psi}_L)$ transform under $N(L)/L$ as L_s under conjugation in $N(L)$:

$$g_s(\eta\underline{\psi}_L)=g_{s(\eta)}(\underline{\psi}_L), \quad L_{s(\eta)} \equiv \eta^{-1}L_s\eta, \tag{8}$$

where η is an element of $N(L)$. Therefore, g_s transform *reducibly* under $N(L)/L$. Its components which belong to different conjugacy classes $[L_s]$ of subgroups L_s transform independently. Each conjugacy class $[L_s]$ gives rise to an exactly $|N(L)|/|N(L) \cap N(L_s)|$-dimensional permutation representation of $N(L) \cap N(L_s)$ in $N(L)$.

In order to remove solutions Eq. (5), $s=i(L)+1,\ldots,m$, from Eq. (7) we can now use again a projection technique; cf. Eq. (8). For example, an equation

$$[1-\mathcal{P}(L_t)] \circ g_s(\underline{\psi}_L) \equiv g_s(\underline{\psi}_L) - |L_t \cap N(L)|^{-1} \sum_{\eta \in L_t \cap N(L)} g_{s(\eta)}(\underline{\psi}_L)=0 \tag{9}$$

must contain Eq. (5) (more precisely, its associated single linear equation) as a factor. Although, two different L_t and L_p may lead to the same equation, all the factors may be removed. The remaining equations will be invariant under $N(L)$. Consequently, they will be functions of an integrity basis of $N(L)$.

With this the symmetry content of Eq. (1) has been exhausted. To summarize the method: The solutions of Eq. (1) are found by exploring subspaces fix$L$ associated with little groups L of R; solutions associated with little groups L_s, $L_s \supset L$, can always be removed from the equations, reducing their degree.

As an illustration, let us consider R to be the cubic group O_h (a more detailed calculation will be presented elsewhere[8]). I consider the Landau free energy of the 6th degree:

$$F(\underline{\psi}) = a\theta_0 + b\theta_0^2 + c\theta_1 + d\theta_0^3 + e\theta_0\theta_1 + f\theta_2, \qquad (10)$$

where $\theta_0, \theta_1, \theta_2$ form an integrity basis for the ring of invariant polynomials of O_h, $\theta_0 = \psi_1^2 + \psi_2^2 + \psi_3^2$, $\theta_1 = \psi_1^4 + \psi_2^4 + \psi_3^4$, $\theta_3 = \psi_1^6 + \psi_2^6 + \psi_3^6$, and $a, \ldots, f$ are functions of thermodynamic variables only. Equation (1), in a component form, reads

$$f_i(\psi) = \sum_{\alpha=0}^{2} A_\alpha(\theta)\psi_i^{2\alpha+1} = 0, \qquad (11)$$

which is a system of three quintic equations in three variables. The quantities A_α depend only through the θ's on $\underline{\psi}$.

A list of little groups of O_h is known.[9] Let us consider a little group C_s, which consists of the identity element and a diagonal reflection plane perpendicular to the $(1, \bar{1}, 0)$ direction. A projector $\mathcal{P}(C_s)$ is, in matrix form,

$$\mathcal{P}(C_s) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad (12)$$

and the equation of the plane fixC_s is [cf. Eq. (3)] $\psi_1 = \psi_2$ which we use to parametrize the plane fixC_s as $(\psi_1, \psi_2, \psi_3) = (x, x, z)$. The normalizer of this plane is D_{2d} with twofold axes along the (1,1, 0) and $(1, \bar{1}, 0)$ directions.

Little groups which contain C_s and which have $i(L_s) = 1$ are C_{3v}, C_{3v}', C_{4v}, C_{2v} with the rotation axes in the (1,1,1), $(1,1,\bar{1})$, (0,0,1), (1,1,0) directions, respectively. The associated projection operators are easily constructed and Eqs. (5) reduce to $x = z$, $x = -z$, $x = 0$, $z = 0$, respectively. Equation (6) leads to functions $g_1(x,z)$ to $g_4(x,z)$ which transform like $(x-z)^2$, $(x+z)^2$, x^2, and z^2, respectively. These transformation properties I use first to choose which two of the g's are most suitable to be kept.

D_{2d} acts on fixC_s reducibly by changing independently z to $-z$ and x to $-x$. Under this action g_3 and g_4 remain invariant, whereas g_1 and g_2 interchange when either x or z changes sign. This was expected anyhow since g_1 and g_2 correspond to C_{3v} and C_{3v}' from the same conjugacy class. Therefore, we keep g_1 and g_2. The group $C_{4v} \cap D_{2d}$ acts on fixC_s by changing x to $-x$ which is equivalent to interchanging g_1 and g_2. As a consequence [cf. Eq. (9)] we find

$$[1 - \mathcal{P}(C_{4v})] \circ g_1 = \tfrac{1}{2}(g_1 - g_2) = xz[A_1 + A_2(x^2 + z^2)] = 0, \qquad (13)$$

which indeed contains x as a factor. Similarly, the group C_{2v} leads to the same equation, giving a factor z. Consequently, we remove these factors from Eq. (13) and we keep another independent equation, for example $\frac{1}{2}(g_1 + g_2) = 0$. In this manner, the original quintic equations in three unknowns are reduced to a system of one quadratic and one quartic equation in two unknowns. Since x^2 and z^2 form an integrity basis for D_{2d} the equations are actually linear and quadratic in x^2 and z^2 and can be solved analytically.

Table I lists all the solutions of Eq. (11). We note that we find solutions corresponding to spontaneous polarization in the reflection planes. However, even at the degree 6 of the potential there is no solution at the completely asymmetrical point, in agreement with a general theorem.[10] Also, I do not discuss here conditions on the reality and stability of the solutions found.

In order to emphasize a connection between Landau and Higgs problems I note that for $d = e = f = 0$ the present $F(\underline{\psi})$, Eq. (10), is identical to the Higgs potential for the SO(7) adjoint representation. In this case symmetry may break from SO(7) (O_h) to SO(5)⊗ U(1) (C_{4v}) or SU(3)⊗ U(1) (C_{3v}) but not to SO(3)⊗SU(2)⊗ U(1) (C_{2v}), which are the maximal little groups.

It is clear that this method is easily generalizable. For example, the method is actually independent of whether R is irreducible or not and whether R is finite or continuous. Furthermore, the method may be easily generalized to the case when f transforms as a tensor, not necessarily a vector, under R; and the origin of the equations is irrelevant.

If we are interested in the solutions of particular symmetry L (only one of the little groups in $[L]$ needs to be examined), this method is perfectly suitable to such a task since it actually never

TABLE I. All 125 solutions of Eq. (11) $x=\pm[-\beta/2\gamma\pm(\beta^2/4\gamma^2-\alpha/\gamma)^{1/2}]^{1/2}$. For each conjugacy class of little groups $[L]$, a typical solution and the total number of solutions n_s are given.[a]

$[L]$	(ψ_1,ψ_2,ψ_3)	$\alpha;\beta;\gamma$	n_s
$[O_h]$	$(0,0,0)$	$\cdots$	1
$[C_{4v}]$	$(x,0,0)$	$2a;\ 4b+4c;\ 6d+6e+6f$	12
$[C_{2v}]$	$(x,x,0)$	$2a;\ 8b+4c;\ 24d+12e+6f$	24
$[C_{3v}]$	(x,x,x)	$2a;\ 12b+4c;\ 54d+18e+6f$	16
$[C_s]$	(x,x,z), $z=\pm[(2c+hx^2)/g]^{1/2}$	$2a-8bc/g+(8ec^2+24dc^2)/g^2;$ $4c+12bf/g+(16ce^2-72cdf)/g^2;$ $10e+6f+(54df^2+8e^3)/g^2$	48
$[C_s']$	$(x,y,0)$, $y=\pm(-x^2-2c/g)^{1/2}$	$2a-8bc/g+(8ec^2+24dc^2)/g^2;$ $4c;\ 4e+6f$	24
$[C_1]$	$\cdots$	$\cdots$	0

[a] $g=2e+3f;\ h=4e+3f$.

requires knowledge of other solutions. For example, one may search for solutions only among maximal little groups L.[2,3,11] In case one is interested in all the solutions, it is most economical to search for them in descending fashion (with respect to subgroup relationship among little groups) until the number of solutions saturates the total number of possible solutions.

In some cases we are also interested in the stability of the solutions found. It is usually determined from positivity of the operator $\partial f(\underline{\psi}^*)$, where $\underline{\psi}^*$ is a solution. It is then obvious that it suffices to look for the stability of $\underline{\psi}^*$ separately in subspaces fix$L$, associated with $\underline{\psi}^*$, and in their respective orthogonal complements in $\mathbb{R}^n$.

The power of this method is most striking in the cases where geometrical intuition fails. For example, even in the case of O_h, Eqs. (11) fail to be geometrically transparent as soon as they are written in an arbitrary coordinate setting. However, the present method is independent of such details. Its strength is even more obvious in cases when $n>3$. Such is the case of a six-dimensional, X-point irreducible representation of the space group $O_h{}^3$, which I will treat, using this method, elsewhere.

I am grateful to Professor L. Michel and Professor R. T. Sharp for reading of the manuscript and for constructive comments. I also acknowledge receipt of an Alexander von Humboldt Research Fellowship.

[1] See, for example, a short review by M. V. Jarić, to be published.

[2] L. Michel, CERN Report No. Th. 2716, 1979 (unpublished).

[3] M. V. Jarić, Phys. Rev. B 18, 2237 (1978), and in *Group Theoretical Methods in Physics*, Lecture Notes in Physics, Vol. 94 (Springer-Verlag, New York, 1979), p. 83.

[4] M. Abud and G. Sartori, Phys. Lett. 104B, 147 (1981); J. Kim, Nucl. Phys. B196, 285 (1982); M. V. Jarić, to be published.

[5] L. Michel, in Proceedings of the 1981 Primorsko Summer School (to be published).

[6] L. Michel, Rev. Mod. Phys. 52, 617 (1980).

[7] M. V. Jarić, Phys. Rev. B 23, 3460 (1981), and to be published.

[8] M. V. Jarić, Institut des Hautes Etudes Scientifiques Report No. IHES/P/82/9 (to be published).

[9] L. Michel and J. Mozrzymas, in *Group Theoretical Methods in Physics*, Lecture Notes in Physics, Vol. 79 (Springer-Verlag, New York, 1978), p. 447.

[10] M. V. Jarić, in *Group Theoretical Methods in Physics*, Lecture Notes in Physics, Vol. 135 (Springer-Verlag, New York, 1980), p. 12.

[11] E. Ascher, Phys. Lett. 20, 352 (1966).