[13]G. D. Boyd and J. P. Gordon, Bell Syst. Tech J. 40, 489 (1961).
[14]S. D. Durbin, S. M. Arakelian, and Y. R. Shen, Opt. Lett. 6, 411 (1981); M. LeBerre, E. Ressayre, and A. Tallet, Phys. Rev. A 25, 1604 (1982), and references therein.

Instability Cascades, Lotka-Volterra Population Equations, and Hamiltonian Chaos

G. Picard and T. W. Johnston
Institut National de la Recherche Scientifique-Energie, Varennes, Québec J0L 2P0, Canada
(Received 18 February 1982)

Parametric decay instability cascades with one wave heavily damped are examples of Lotka-Volterra time-dependent equation systems. These are shown to be constant Hamiltonian in nature and the cascades (at least) can display Hamiltonian chaos, even without ensemble averaging. It seems likely that sufficiently complicated systems of this type will usually be chaotic unless very near equilbrium. These conclusions apply as well to some chemical and population biology systems which obey similar equations.

PACS numbers: 52.35.Mw, 03.40.-t, 87.10.+e

When an instability mode which is driven unstable can have its growth reversed because it is depleted by driving (pumping) another similar mode via a three-mode interaction, and when that process can be repeated over several pump modes, one can speak of an instability *cascade*. If the modes that are not acting as pumps are heavily damped (as in nonlinear Landau damping for plasma waves), the relative phases have no interesting behavior and the equations for the mode actions are a particular case of the general Lotka-Volterra interaction system, that of a nearest-neighbor cascade.[1] (This can be thought of as a system of predation where each population species preys only on the population next nearest the ultimate source and is being preyed on only by the population next further from the source.)

The general Lotka-Volterra equation system[2] is as follows, for the variables n_i, their natural linear growth rates k_i, and interaction parameters a_{ij} and β_i:

$$dn_i/dt \equiv \dot{n}_i = k_i n_i + \sum_j a_{ij} n_i n_j/\beta_i. \qquad (1)$$

Here n_i (which must be positive to be acceptable) can represent the ith concentration of, for instance, species, chemical component, or wave action. The a_{ij} is antisymmetric, reflecting the zero-sum nature of the ij pair interactions, while the β_i coefficients (which are positive) are the Volterra species equivalence numbers. For chemistry the β_i coefficients are such that β_i/β_j is the ratio of the number of molecules involved in the ij interaction, and for three-wave interactions all the β_i coefficients are 1 (from the Manley-Rowe relations). The original two-species version of these equations was invoked by Lotka[2] for autocatalytic reactions and by Volterra[2] to explain fluctuations in fish species in the Adriatic sea, while several authors have used them to consider plasma-wave instability competition.[1]

On numerical investigation, it became evident that the time behavior could be remarkably irregular. Using Poincaré surfaces of section we found the behavior to resemble that of Hamiltonian chaos,[3] and then realized that the system was indeed Hamiltonian. We recapitulate for convenience an earlier treatment[2] of an even number of equations, with a slight (but important) variation, because this leads naturally to the Hamiltonian. We define

$$w_i \equiv \beta_i \ln(n_i/|\gamma_i|),$$

where γ_i is the formal equilibrium value of n_i. The γ_i hence must satisfy Eq. (1) with $\dot{n}_i$ set equal to zero.

For an odd number of equations the determinant of the antisymmetric a_{ij} is necessarily zero, which is why we consider only even systems with $2T$ members and invertible a_{ij}'s. Note that, unlike n_i's, some γ_i's may be negative. (We return to this point later when discussing nearest-neighbor cascades.) It has been shown[2] that there is a general Lotka-Volterra invariant, which we call H because it is in truth a Hamiltonian, given by

$$H = \sum_i \beta_i (n_i - \gamma_i \ln n_i + \gamma_i \ln|\gamma_i|). \qquad (2)$$

We have used w_i here (rather than the earlier[2] w_i/β_i) because this allows the following attractive form for the equations when H is expressed

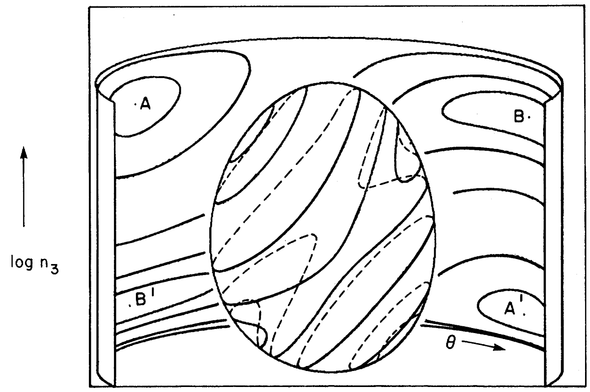

FIG. 1. Sketch showing projection used in Fig. 2 of Poincaré surface of section for $N=2$ $(n_4=\langle n_4\rangle)$ in n_1-n_2-n_3 space on to a cylinder $\{z=\log_{10}n_3,\ \theta=\tan^{-1}[-(n_1-\gamma_1)/(n_2-\gamma_2)]\}$. Note that A,A' and B,B' are corresponding regions with trajectories piercing each alternately.

in terms of w_i:

$$\dot{w}_i=\sum_j a_{ij}\,\partial H/\partial w_j\,. \tag{3}$$

The Liouville theorem is satisfied,[2] an encouragement to apply the Darboux theorem,[4,5] which proves that the system is Hamiltonian.

Let us now turn to the cascade system. For the wave problems of interest to us β_i is 1, but even for other cascade problems, where other values of β_i apply, this would only change quantitative aspects. The same remark applies roughly to a_{ij}, whose magnitude we take for convenience to be 1 for all equations, so that we now have for $2T$ equations $a_{ij}=\delta_{i,j+1}-\delta_{i,j-1}$, giving

$$\begin{aligned}\dot{n}_s&=k_sn_s+n_{s-1}n_s-n_sn_{s+1}\\&=n_s(n_{s-1}-\gamma_{s-1})-n_s(n_{s+1}-\gamma_{s+1})\,.\end{aligned} \tag{4}$$

(We take n_s, γ_s as formally zero for $s<1$, $s>2T$.)

With the definition $L_s=\ln n_s$, the equations for L_s follow immediately from Eq. (4). Note that

$$\gamma_{s+1}-\gamma_{s-1}=k_s\,. \tag{5}$$

This allows easy calculation of γ_s by recursion.

While the canonical variables can be obtained

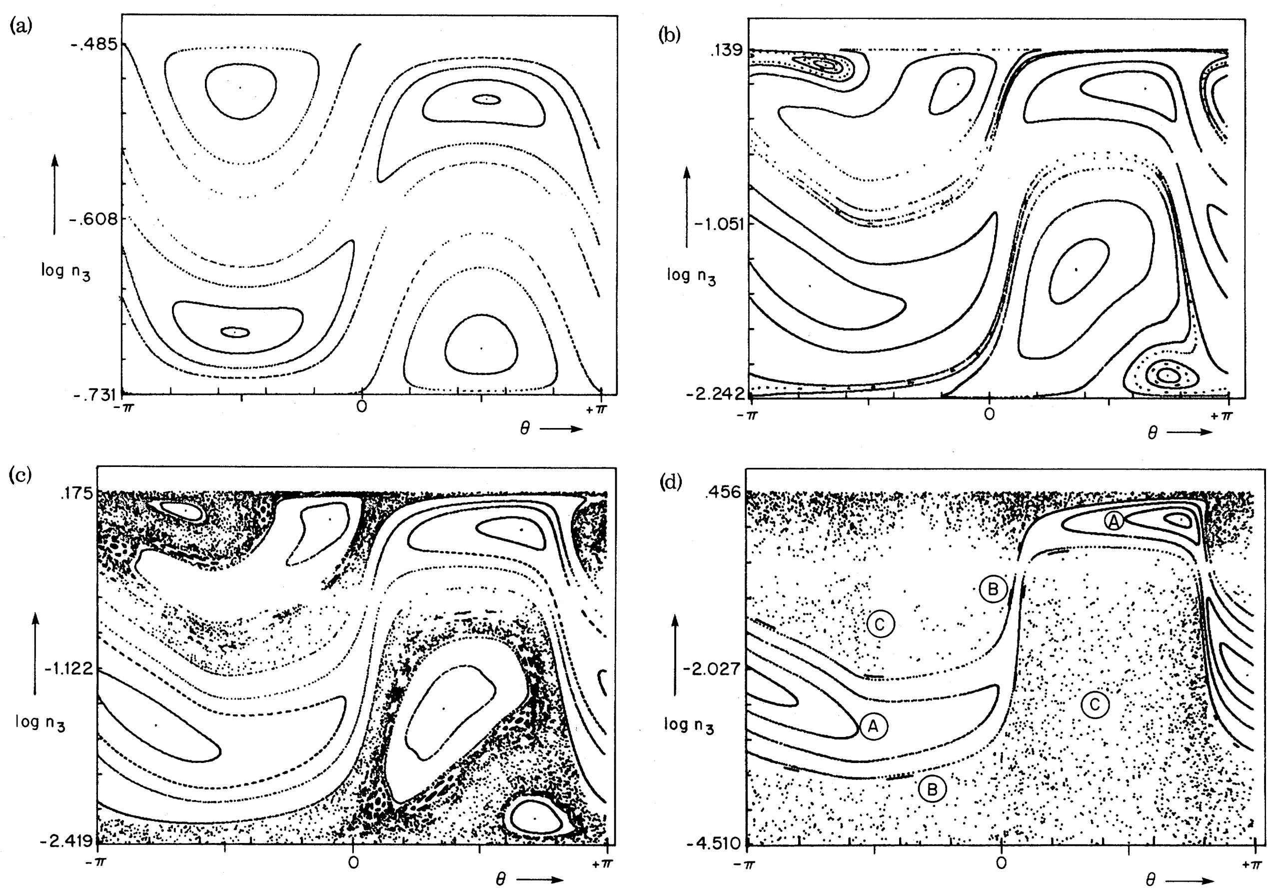

FIG. 2. Projection (see Fig. 1) of Poincaré surfaces of section for $H-H_{\mathrm{eq}}$ values of (a) 0.01, (b) 0.7, (c) 0.8, and (d) 2.0.

by the Darboux procedure, the pattern soon becomes clear after several values of T. The general cascade result was easy to guess and to verify and its most convenient form proves to be

$$q_r = L_{2r}, \; p_r = \sum_{s=1}^{r} L_{2s-1}.$$

The canonical equations of motion follow immediately when H is written in terms of q_r and p_r.

A point left undiscussed is that of equilibrium values γ_s and what happens if they are negative. Space does not permit a general discussion, but we have satisfied ourseleves that for the interesting cascade case (k_s positive for $s=1$ and negative for $s>1$) the system will evolve to an even system with all equilibrium values γ_s positive. Providing only that a sufficient number of equations is included, any excess n_s for s greater than some even value will decrease to zero for initially even[2] or odd[6] systems.

We have calculated some particular cases for $N=2$ (i.e., four equations) quite extensively, and will now present some results. We take a common damping ν equal to 0.25 and a pump of strength 1, so that k_1 is $1-0.25$ and the other k's are -0.25, and hence γ_s values are, in order, 0.5, 0.75, 0.25, and 0.5.

Let us look first at the two-dimensional Poincaré surfaces of section for n_4 equal to γ_4. (Since γ_4 is also the average value[2] of n_4 it gives the most representative section.) This closed surface is simply connected in n_1-n_2-n_3 space and we use a cylindrical projection as shown in Fig. 1 to display the results in Fig. 2. Close to equilibrium the surface of section is ellipsoidal and the points from a given trajectory lie on the intersection with another set of ellipsoids (quasi-invariant near equilibrium) as shown in Fig. 2(a). These are of course Kolmogorov-Arnold-Moser (KAM) tori[4] sections. (Like darning a sock, a trajectory alternates, passing one way then the other through a surface of section, usually going

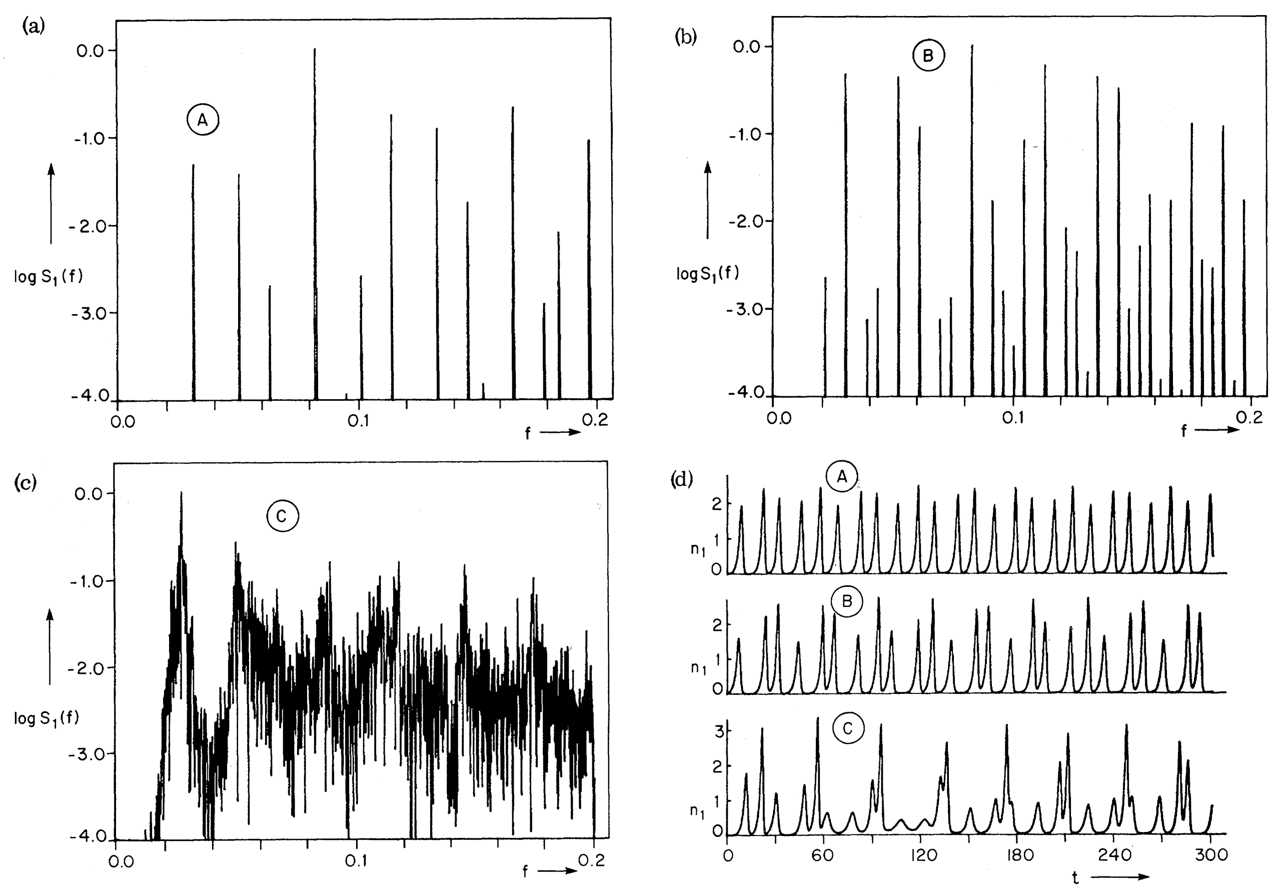

FIG. 3. Typical frequency spectra (a)–(c) and time behavior (for n_1) (d) from the orbits of Fig. 2(d) which are indicated by A, B, and C, respectively.

alternatively from one curve to another.)

For rather larger values of H, equivalent to starting farther from equilibrium, a pair of "tear drops" form, as shown in Fig. 2(b), and with them a chaotic belt. This rapidly develops the now-classic microisland structure ("Polynesia" or "micronesia"), and then [Fig. 2(c)] the "stochastic sea" becomes evident and begins [Fig. 2(d)] to drown the "coherent continents."

While the general progression is typical of Hamiltonian systems,[7] there is a significant difference from other Hamiltonian systems with interacting oscillators. Here the rapid growth of chaos is linked to the tear-drop formation rather than the resonance overlap.[7] We have as yet no useful criterion to predict this onset.

It is instructive to look at the spectra and time behavior and so, in Fig. 3, they are shown for various trajectories of Fig. 2(d) as indicated by the letters. One sees the two-frequency system expected with two degrees of freedom, and as one approaches chaos the beat products become more prominent. We note without comment the qualitative resemblance between the spectra of Fig. 3 and those of Fig. 1 of Gollub and Swinney[8] for circular Couette flow.

Some implications of this work are considered next. We have shown that the Lotka-Volterra equations are in fact Hamiltonian and can display (at least in our example) chaotic behavior similar to (but not identical with) that of other nonlinear systems with a constant Hamiltonian. It may well be that such chaos is likely in many Lotka-Volterra systems of sufficient complexity ($T \geq 2$), and perhaps more likely the larger T. Goel, Maitra, and Montroll[2] discussed at length the ensemble average of Lotka-Volterra systems (and others). It seems likely that many such systems can, via chaotic behavior, be nearly ergodic without the necessity of ensemble averaging. [We say "nearly ergodic" because quite significant phase regions may be avoided by the chaotic trajectories, as in Fig. 2(d).]

[1] B. Kadomtsev, *Plasma Turbulence* (Academic, New York, 1965), p. 65; R. Z. Sagdeev and A. A. Galeev, *Nonlinear Plasma Theory* (Benjamin, New York, 1969), p. 98; A. Hasegawa, *Plasma Instabilities and Nonlinear Effects* (Springer-Verlag, New York, 1975), pp. 187–189; R. B. White, Y. C. Lee, and K. Nishikawa, Phys. Rev. Lett. 29, 1315–1318 (1972); W. L. Kruer and E. J. Valeo, Phys. Fluids 16, 675 (1973); A. Hasegawa and L. Chen, Phys. Fluids 18, 1321–1326 (1975); S. L. Musher and A. M. Rubenchik Fiz. Plazmy 1, 982 (1975) [Sov. J. Plasma Phys. 1, 536 (1975)]; F. Y. F. Chu and C. F. F. Karney, Phys. Fluids 20, 1728–1732 (1977).

[2] N. S. Goel, S. C. Maitra, and E. W. Montroll, Rev. Mod. Phys. 43, 231–276 (1971), also published as *On the Volterra and Other Nonlinear Models of Interacting Populations* (Academic, New York, 1971).

[3] Useful summaries exist in J.-P. Eckmann, Rev. Mod. Phys. 53, 643–654 (1981); E. Ott, Rev. Mod. Phys. 53, 655–671 (1981).

[4] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1978), Chap. 8, Sec. 43; R. G. Littlejohn, J. Math. Phys. (N.Y.) 20, 2445 (1979).

[5] R. G. Littlejohn, Phys. Fluids 24, 1730–1749 (1981).

[6] B. Rosen, T. Roycraft, and G. Schmidt, Phys. Fluids 20, 1104–1108 (1977).

[7] B. V. Chirikov, Phys. Rep. 52, 263–379 (1979), and in *Topics in Nonlinear Dynamics—1978*, edited by S. Jorna, AIP Conference Proceedings No. 46 (American Institute of Physics, New York, 1978), and in *Intrinsic Stochasticity in Plasmas*, edited by G. Laval and D. Gresillon (Les Editions de Physique Courtabouef, Orsay, France 1979); *Nonlinear Dynamics*, edited by R. H. G. Helleman (New York Academy of Sciences, New York, 1980); G. H. Walker and J. Ford, Phys. Rev. 188, 416–432 (1969).

[8] J. P. Gollub and H. L. Swinney, Phys. Rev. Lett. 35, 927–930 (1975).