and

$$\mu = Z\mu_p' = 1.0011 m_e.$$

$H_3(r,\theta;R)$ is essentially the same as the Hamiltonian $H_2(r,\theta;R)$ of I; it differs from it only in that m_e is replaced by μ. This makes it possible to use a lower bound potential already calculated.

It can be shown by solving the appropriate radial Schrödinger equation that $V_L(R)$ cannot support a bound state and consequently no bound state of $H_1(R)$ exists. It follows that no bound state of $H_{\mathrm{inf}'}$ exists. Hence H_{fin}, the Hamiltonian for the internal motion of $e^+\mathrm{H}$ in the case of finite proton mass, can have no bound state. This proves, for the first time, that no bound state of $e^+\mathrm{H}$ exists in the case when the finite mass of the proton is taken into account.

It is intended as a next stage to apply the method to try to prove the absence of a bound state for a system made up of muonium and a positron. The mass of the muon is $207 m_e$ and, consequently, the effect of the Hughes-Eckart term can be expected to be more important in this case.

I wish to thank Dr. M. Farid, and Dr. D. C. Mattis of the University of Utah for drawing my attention to the form of the Hamiltonian for the internal motion of e^+H with origin at the electron.

[1]I. Aronson, C. J. Kleinman, and L. Spruch, Phys. Rev. A 4, 841 (1971).
[2]E. A. G. Armour, J. Phys. B 11, 2803 (1978).
[3]E. A. G. Armour and D. M. Schrader, to be published.
[4]D. S. Hughes and C. Eckart, Phys. Rev. 36, 694 (1930).
[5]M. Farid and D. C. Mattis, private communication.
[6]L. Spruch, in *Atomic Collision Processes*, edited by S. Geltman, K. T. Mahanthappa, and W. E. Brittin, Lectures in Theoretical Physics, Vol. 11 (Gordon and Breach, New York, 1969), p. 65.
[7]H. Wind, J. Chem. Phys. 42, 2371 (1965).

Quantum Mechanical Models of Turing Machines That Dissipate No Energy

Paul Benioff
Division of Environmental Impact Studies, Argonne National Laboratory, Argonne, Illinois 60439
(Received 22 March 1982)

Quantum mechanical Hamiltonian models of Turing machines are constructed here on a finite lattice of spin-$\frac{1}{2}$ systems. The models do not dissipate any energy and they operate at the quantum limit in that the system (energy uncertainty)/(computation speed) is close to the limit given by the time-energy uncertainty principle.

PACS numbers: 03.65.-w, 05.30.-d

There has been much discussion in the literature recently about the physical limitations of the computation process and information transfer.[1-12] Early work[1,2] assumed that the computation process was irreversible because of the necessity to discard information. However, this was shown to be false by Bennett.[3] More recently Landauer[7] has stressed the importance of the energy dissipation problem for the computation process. Recent work[8,9] has assumed that energy dissipation must occur, and on the basis of this, Bekenstein[8] derived an upper limit of 10^{15} steps/sec for computation speeds. However, this work has been criticized by Deutsch[10] and Landauer.[11] Fredkin and Toffoli[12] have constructed a classical mechanical model of the computation process which dissipates no energy.

The purpose of this note is to briefly present and discuss quantum mechanical Hamiltonian models of the computation process as represented by standard Turing machines.[3] These models also dissipate no energy and operate at essentially the quantum limit in that the total system (energy uncertainty)/(computation speed) $\lesssim 2\pi\hbar$. Unlike the models constructed in other work,[13] the models constructed here do not use successive scattering to drive the process.

A Turing machine consists of three parts, an internal machine part L which is capable of assuming any one of a finite number of states, a computation tape T infinite in both directions, and a computation head h. T is divided into an infinite number of cells at positions $\ldots -1, 0, 1, \ldots$. Each cell contains any symbol s of an

alphabet S which includes the blank. h scans one cell at a time. In one computation step, h changes s to some s' in S and shifts one cell to the right or to the left or remains where it is. The state of L can also be changed, and $s=s'$ is possible.

Each Turing machine Q corresponds to a map $\tau_Q: N\times S \to N\times S\times\{-1, 0, 1\}$, where N is the set of possible L states and $-1, 1, 0$ refer to left, right, or no shift of h. A computation is described by iterations of τ_Q on $N\times S$. That is, if for some step, h is at position j and $\tau_Q(ls)=(l's'\alpha)$, the next step is given by $\tau_Q(l's_1)$ where s_1 is the symbol scanned by h at position $j+\alpha$. A computation halts if L is in state l, h scans symbol s, and $\tau_Q(ls)=(ls0)$.

Turing machines are very slow but very powerful in that any computation made on any digital machine can be done on a Turing machine. Turing machines are also computation universal. That is, there exists a Turing machine Q_u such that Q_u, given both the name G_Q (suitably encoded) of any Turing machine Q and some expression γ on T, carries out the same computation on γ as G_Q does.

Turing machine computations are not in general reversible. However, it was shown by Bennett[3] that each Turing machine can be extended to a Turing machine whose computations are reversible by generating a history of the computation. Here the model is extended by addition of a record system R containing, initially, blank record cells at positions $0, 1, \ldots$ and a record head i initially at position 0. A filled record cell contains the triple (lsj) where l is in N, s is in S, and j denotes the position of h. As the computation progresses a history as a sequence φ of (lsj) values will be generated in R as i is shifted along R.

From now on the concern here will be with reversible standard computations. A standard computation begins with L in a designated state, "1", h and i at positions 0, and all cells in T to the left of 0 blank. Also, at most a finite number of cells of T are nonblank and no two nonblank cells are separated by one or more blanks. The final state is similar except that L is another special state.

Each computation of a step of the original machine Q corresponds here to a record, compute, and i-shift operation in the extended reversible machine. In the record operation, the state of L, the symbol in the cell scanned by h, and the position of h are recorded into the blank record cell scanned by i. In the compute operation, the (lsj) triple recorded in the R cell scanned by i and the value of τ_Q at (ls) give the changes to be made in the state of L, the symbol in the cell scanned by h, and the position of h. Finally, i is shifted to a fresh record cell.

The overall state of a reversible machine at the end of a computation step is given by some $(l\lambda jk\varphi)$ where l denotes the state of L, j and k are the positions of h and i, respectively, and λ and φ are expressions on T and R, respectively. Let C be the set of all such descriptions of all reversible machines.

Each machine computation corresponds to iterations of a bijection $T_Q: C\to C$ such that for each $(l\lambda jk\varphi)$ for which $\varphi(k)=b$, $T_Q(l\lambda jk\varphi)=(l'\lambda'j'k'\varphi')$. Here $\varphi'=\varphi$ except at cell k of R where $\varphi'(k)=[l, \lambda(j), j]$, $k'=k+1$, $\lambda'=\lambda$ except at cell j of T, $j'=j+\alpha$, and $\tau_Q[l, \lambda(j)]=[l', \lambda'(j), \alpha]$.

The detailed construction of the maps T_Q is described elsewhere.[14] Here one notes that for each standard initial state $(1\gamma 00b)$, where L is in state 1, γ is the initial expression on T, h and i are at position 0, and the R cells are all blank, one has an orbit $\{T_Q{}^n(1\gamma 00b)\,|\,n=0, 1, \ldots\}$ which is a T_Q-invariant irreducible subset of C. The orbit is finite if and only if the computation halts. T_Q = identity outside of the orbits.

From now on, to avoid technical complications consideration will be limited to a finite but arbitrary number J of computation steps of any standard machine. Then N is finite, T extends from cell $-J$ to cell J with $2J+1$ cells, R extends from 0 to J with $J+1$ cells, in the definition of T_Q one has $k'=k+1 \bmod(J+1)$ and $j'=j+\alpha$ [$\bmod(J+1$ or $-J-1)$], and C becomes a finite set. For each γ the orbit of T_Q at $(1\gamma 00b)$ becomes a finite set with N_γ elements.

Consider a system with a set of quantum states which are in one-to-one correspondence with the $(l\lambda jk\varphi)$ in C. How the correspondence is constructed is of no concern at present. However, these states $\Psi_{l\lambda jk\varphi}$ must be orthonormal. Let $\mathcal{H}$ be the finite-dimensional Hilbert space spanned by $\{\Psi_{l\lambda jk\varphi}\,|\,(l\lambda jk\varphi)\in C\}$.

For each machine Q define an operator V_Q on $\mathcal{H}$ by

$$V_Q\Psi_{l\lambda jk\varphi}=\Psi_{T_Q(l\lambda jk\varphi)} \tag{1}$$

for all $l\lambda jk\varphi$. V_Q is unitary since T_Q is a bijection. Corresponding to the orbits of T_Q are the orbits $\{\Psi_n{}^\gamma\,|\,n=0, 1, \ldots, N_\gamma-1\}$ of V_Q each of which span a subspace $\mathcal{H}_\gamma$ of $\mathcal{H}$. Here $\Psi_n{}^\gamma=V_Q{}^n\Psi_0{}^\gamma$, where $\Psi_0{}^\gamma=\Psi_{1\gamma 00b}$. Each $\mathcal{H}_\gamma$ is V_Q invariant and

irreducible. Thus one has $V_Q=\sum_\gamma V_\gamma P_\gamma+1-\sum_\gamma P_\gamma$ (the Q subscript is suppressed from now on) where P_γ projects onto $\mathcal{H}_\gamma$.

Let Δ be an arbitrary fixed time interval. One now requires that for each Q a Hamiltonian H exists such that one has $V_Q=\exp(-iH\Delta/\hbar)$. With $H=\sum_\gamma H_\gamma$, where H_γ acts only inside $\mathcal{H}_\gamma$, one has

$$\exp(-iH\Delta/\hbar)=\sum_\gamma \exp(-iH_\gamma\Delta/\hbar)P_\gamma+(1-\sum_\gamma P_\gamma). \quad (2)$$

Since $V_\gamma=\exp(-iH_\gamma\Delta/\hbar)$ is the bilateral shift on $\mathcal{H}_\gamma$, or $V_\gamma\Psi_n{}^\gamma=\Psi^\gamma_{n+1 \bmod N_\gamma}$ and $\mathcal{H}_\gamma$ is finite, V_γ is pure discrete. By standard procedures one can find eigenvalues and eigenvectors of V_γ and obtain a Hamiltonian H_γ which satisfies the above. A convenient choice is

$$H_\gamma=\sum_{l=0}^{N_\gamma-1}(2\pi l\hbar/N_\gamma\Delta)Q_l{}^\gamma,$$

where $Q_l{}^\gamma$ is the projection operator for the eigenvector

$$\Phi_l{}^\gamma=N_\gamma{}^{-1/2}\sum_{j=0}^{N_\gamma-1}(a_l)^{1-j}\Psi_j{}^\gamma$$

with associated eigenvalue $a_l=\exp(-2\pi il/N_\gamma)$ of V_γ.

It is useful to express H_γ and $V_\gamma(t)=\exp(-iH_\gamma t/\hbar)$ in terms of the original basis $\Psi_n{}^\gamma$. By standard techniques one has

$$H_\gamma=\sum_{j,k=0}^{N_\gamma-1}d_{jk}\sigma_{jk}{}^\gamma P_k{}^\gamma, \quad (3)$$

where $P_k{}^\gamma$ is the projection operator for $\Psi_k{}^\gamma$, $\sigma_{jk}{}^\gamma$ exchanges $\Psi_j{}^\gamma$ and $\Psi_k{}^\gamma$, and the coefficient d_{jk} is given by

$$d_{jk}=\sum_{l=0}^{N_\gamma-1}\frac{2\pi l\hbar}{\Delta N_\gamma{}^2}\exp[2\pi il(j-k)/N_\gamma]. \quad (4)$$

$V_\gamma(t)$ is given by

$$V_\gamma(t)=\sum_{j,k=0}^{N_\gamma-1}b_{jk}(t)\sigma_{jk}{}^\gamma P_k{}^\gamma, \quad (5)$$

where the coefficient $b_{jk}(t)$ is given by

$$b_{jk}(t)=\frac{1}{N_\gamma}\sum_{l=0}^{N_\gamma-1}\exp\left[\frac{-2\pi il}{N_\gamma}\left(\frac{t}{\Delta}+k-j\right)\right]. \quad (6)$$

It is clear that the system Hamiltonian given by Eqs. (2) and (3) has the desired properties. It is time independent. Also the evolution of $\Psi^\gamma(t)=\exp(-iHt/\hbar)\Psi_0{}^\gamma$ is such that at time $t=n\Delta$, $\Psi^\gamma(n\Delta)=\Psi^\gamma_{n \bmod N_\gamma}$. This follows from the fact that $b_{jk}(n\Delta)=0$ unless $(n \bmod N_\gamma+k-j)=0$ or N_γ. For both of these cases $b_{jk}(n\Delta)=1$. $t=N_\gamma\Delta$ is the recurrence cycle time for any state in $\mathcal{H}_\gamma$.

An example of the above can be constructed by letting the quantum system be a finite two-dimensional quantum lattice of spin-$\frac{1}{2}$ systems in which separate disjoint regions of the lattice correspond to the subsystems L, T, h, i, and R of the reversible machine. Each state $\Psi_{l\lambda jk\varphi}=\Psi_l{}^L\otimes\Psi_\lambda{}^T\otimes\Psi_j{}^h\otimes\Psi_k{}^i\otimes\Psi_\varphi{}^R$ describes a configuration of spin projections along some fixed direction for each spin in the lattice, and C becomes the set of all lattice configurations. Details of the correspondence of $l\lambda jk\varphi$ to $\Psi_{l\lambda jk\varphi}$ which involves constructing binary representations for the states of L, expressions on T and R, and positions of h and i are given elsewhere.[14] Figure 1 gives some details.

For any configuration f on all or part of the spin lattice the projection operator P_f for finding configuration f is given by $P_f=\otimes_{(ij)\in \mathrm{Dom}f}\times P_{f(ij)}(i,j)$, where Dom$f$ is the lattice region associated with f. The projection operator for finding the spin at site (i,j) up (+) or down (−) is given by $P_\pm(i,j)=[1\pm\sigma_3(i,j)]/2$, where $\sigma_3(i,j)$

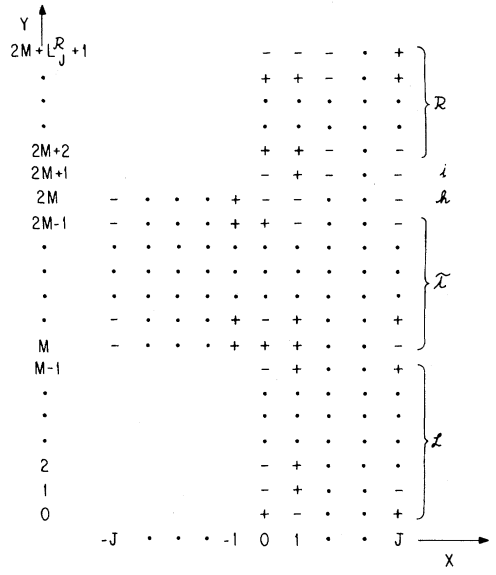

FIG. 1. The lattice model of the overall computing system. M and $L_J{}^R$ are the number of sites needed to represent the number of symbols in S and the number of (lsj) triples reachable in $\leq J$ steps. A plus sign denotes spin up and a minus sign denotes spin down. The dotes indicate that the regions are filled with one spin-$\frac{1}{2}$ system at each site. The brackets give the lattice subregions associated with the parts of the machine.

is the Pauli matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for the spin at (i,j).

For any pair f, g of configurations defined over the same lattice region the exchange operator σ_{fg} is defined by $\sigma_{fg} = \otimes_{(i,j)\in D_{fg}} [\sigma_1(i,j)]$, where D_{fg} is the set of sites at which f differs from g and

$$\sigma_1(i,j) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the spin-flip operator for the spin at site (i,j). Note that $\sigma_{fg}\Psi_h \neq \Psi_h$ for any configuration h whose domain has a nonzero overlap with D_{fg}.

The models constructed here describe the first J steps of any Turing machine computation. If the computation halts in $m < J$ steps then the same final $(l\,s\,j)$ triple is recorded in cells $m, m+1, \ldots, J$ of the record system. The states of L, T, and h are the same at all times $n\Delta$, where $m \leq n \leq J$, as they must be for a completed computation. For times $n\Delta$ with $J < n < N_\gamma$ the states $\Psi^\gamma(n\Delta) = \Psi_n{}^\gamma$ describe a complex inverse phase in which the record is erased and the computation is undone. If desired, the reversal phase can be made to occur as the exact inverse of the forward phase.[3,14]

There is also no state degradation or energy dissipation as the models evolve. At each time $n\Delta$ with $n = 0, 1, \ldots, N_\gamma - 1$ the overall system state $\Psi^\gamma(n\Delta) = \Psi_n{}^\gamma$. It is *not* a linear combination

$$\sum_{m=0}^{N_\gamma - 1} c_m(n\Delta)\Psi_m{}^\gamma$$

of states $\Psi_m{}^\gamma$ where the coefficients $c_m(n\Delta)$ are such that $c_m(n\Delta) \neq 0$ and the amplitude $|c_n(n\Delta)|$ for finding $\Psi_n{}^\gamma$ in $\Psi^\gamma(n\Delta)$ decreases as n increases. This would be the case if the overall system state were degrading as it evolves.

The models constructed here also operate essentially at the quantum limit in that the energy uncertainty δE divided by $1/\Delta$, the computation speed, is close to the limit given by the time-energy uncertainty principle. This follows from the fact that the spread in energy eigenvalues, given by $(2\pi\hbar/\Delta)(1 - N^{-1}) \approx 2\pi\hbar/\Delta$, is an upper limit for δE, whereas the uncertainty principle gives $\delta E \gtrsim \hbar^\gamma/\Delta$.

Also for the models constructed here, limits on computation speed which may arise from energy dissipation[8] are not present. One can in principle at least increase the computation speed by increasing the average system energy $\langle H_\gamma \rangle$ without introducing state degradation and energy dissipation. In particular

$$\langle H_\gamma \rangle = \langle \Psi^\gamma(t) H \Psi^\gamma(t) \rangle = (\pi\hbar/\Delta)(1 - N^{-1}) \approx \pi\hbar/\Delta$$

which is proportional to the computation speed. For example, if the average system energy is 1 eV, the speed is 5×10^{14} steps/sec; if the energy is 1 erg, the speed is 3×10^{26} steps/sec which is greater than Bekenstein's[8] upper limit of 10^{15} steps/sec.

The mathematical existence of quantum mechanical models which dissipate no energy, such as those constructed here, support Deutsch[10] and Landauer[11] in their criticism of Bekenstein,[8] Mundici,[9] and Bremerman.[1] Whether or not such models can actually be physically constructed is, however, an open question.

For times $n\Delta + \delta$ which are not integral multiples of Δ, i.e., $0 \leq \delta \leq \Delta$, the state of the system is obtained from Eq. (5) as

$$\Psi^\gamma(n\Delta + \delta) = V_\gamma(n\Delta + \delta)\Psi_0{}^\gamma = \sum_{m=0}^{N_\gamma - 1} b_{m-(n \bmod N_\gamma)}(\delta)\Psi_m{}^\gamma. \quad (7)$$

Equation (7) shows that at such times the overall model system state is a linear combination over all representation states in $\mathcal{H}_\gamma$. This type of evolution can be called *time global* because the linear combination includes states corresponding to all stages that the system reached in the past and all stages that it will reach in the future.

It will be seen elsewhere[14] that one can also construct time-dependent Hamiltonian models of Turing machines such that the evolution is *time local*. That is, at times which are not integral multiples of Δ, the overall system state is a linear combination of just two representation states, namely those corresponding to the stage the computation has just left and the one to which it is going. For models such as the spin-lattice models, time-global and time-local evolutions are quite different with respect to restrictions which some measurements on the system must satisfy. Details will be given elsewhere.[14]

[1]H. Bremerman, in *Self-Organizing Systems*, edited by M. C. Yovits, G. T. Jacobi, and G. D. Goldstein (Spartan Books, Washington, D.C., 1967), Pt. 1.

[2]R. Landauer, IBM J. Res. Dev. 5, 182 (1961); R. Landauer and J. W. F. Woo, J. Appl. Phys. 42, 2301 (1971); R. W. Keyes and R. Landauer, IBM J.

Res. Dev. 14, 152 (1970).

[3]C. H. Bennett, IBM J. Res. Dev. 17, 525 (1973).

[4]H. J. Bremerman, in Proceedings of the Conference on Physics of Computation, Massachusetts Institute of Technology, 6–8 May 1981 (to be published).

[5]R. Landauer, Ber. Bunsenges. Phys. Chem. 80, 1048 (1976).

[6]K. K. Likharev, "Classical and Quantum Limitations on Energy Consumption in Computation," 1981 (to be published); L. B. Levitin, "Physical Limitations of Rate, Depth, and Minimum Energy in Information Processing" (to be published).

[7]R. Landauer, in Proceedings of the Conference on Physics of Computation, Massachusetts Institute of Technology, 6–8 May 1981 (to be published).

[8]J. D. Bekenstein, Phys. Rev. Lett. 46, 623 (1981).

[9]D. Mundici, Nuovo Ciminto 61B, 297 (1981).

[10]D. Deutsch, Phys. Rev. Lett. 48, 286 (1982).

[11]R. Landauer, Informal notes, 1981 (unpublished).

[12]E. Fredkin and T. Toffoli, Massachusetts Institute of Technology, Technical Memorandum No. MIT/LCS/TM-197, 1981 (unpublished).

[13]P. Benioff, J. Stat. Phys. 22, 563 (1980), and J. Math. Phys. (N.Y.) 22, 495 (1981), and in Proceedings of the Conference on Physics of Computation, Massachusetts Institute of Technology, 6–8 May 1981 (to be published).

[14]P. Benioff, to be published.