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Chaotic Attractors in Crisis

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The occurrence of sudden qualitative changes of chaotic (or "turbulent") dynamics is discussed and illustrated within the context of the one-dimensional quadratic map. For this case, the chaotic region can suddenly widen or disappear, and the cause and properties of these phenomena are investigated.

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In dissipative physical systems, such as occur in plasmas, fluids, acoustics, optical systems, solid-state devices, etc., it is often observed that the system settles into a state of sustained "chaotic" or "turbulent" motion (cf. Refs. 1 and 2 for a partial listing of some recent relevant physical examples). Furthermore, this chaotic behavior is now understood to result from the presence of strange attractors. [A strange attractor may be thought of as a complicatedly shaped surface in the phase space of the dynamical variables, to which the system orbit is asymptotic in time and on which it wanders in a chaotic fashion (cf. Ott[3] for a recent review).] The features of such states have recently been shown to be well described by surprisingly simple nonlinear dynamical models (e.g., the one-dimensional quadratic map to be discussed below). In many experiments, changes in the system behavior are studied as some parameter of the system is varied. Thus, much theoretical interest has focused on characterizing the evolution of the dynamics as a function of a system parameter.[4-9] In this paper we investigate sudden qualitative changes in chaotic dynamical behavior which occur at parameter values at which the attractor collides with an unstable periodic orbit. We call such events *crises*.

In order to fix ideas and provide a clear, simple illustration of the phenomenon in question, we first consider an elementary case involving the one-dimensional map given by

$$x_{n+1} = C - x_n^2 = F(x_n, C). \tag{1}$$

For $C < -\frac{1}{4}$, no fixed point of the map exists, and all orbits are asymptotic to $x = -\infty$. At $C = -\frac{1}{4}$ a tangent bifurcation occurs at which a stable and an unstable fixed point are created. It is well

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known[7,10,11] that as C is increased past $-\frac{1}{4}$, the stable fixed point undergoes period doubling followed by chaos. [For $C>-\frac{1}{4}$, Eq. (1) can be transformed by a change of variables to the logistic map, $x_{n+1}=rx_n(1-x_n)$; note, however, that the logistic map does not possess a tangent bifurcation analogous to that of Eq. (1) at $C=-\frac{1}{4}$ due to its nongeneric behavior at $r=1$.] As C is increased past $C=2$, the chaotic attracting orbit is destroyed, and all initial conditions lead to orbits which approach $x=-\infty$ (corresponding to the logistic map with $r>4$). Figure 1 gives a bifurcation diagram illustrating the above. In this figure we have plotted the position of the unstable fixed point created at $C=-\frac{1}{4}$, $x=-x_*=-\frac{1}{2}-[\frac{1}{4}+C]^{1/2}$, as a dashed curve. For $2\geqslant C\geqslant -\frac{1}{4}$, and for almost any initial point in the range $|x|<x^*$ the orbit generated by (1) is asymptotic to the bounded orbits shown in Fig. 1. Conversely, any point in $|x|>x_*$ generates an orbit which is asymptotic to $x=-\infty$. Thus, for $-\frac{1}{4}\leqslant C\leqslant 2$, the range $|x|<x_*$ is the *basin of attraction* for bounded orbits, while $|x|>x_*$ is the basin of attraction for $x=-\infty$. Note from the figure that destruction of the chaotic orbit at $C=2$ coincides with the *intersection of the chaotic band with the unstable fixed point* $x=-x_*$. To understand why this happens, consider C to be slightly larger than 2. In this case, a typical initial condition in the region which was chaotic for C slightly less than 2 will generate a chaotic-looking orbit (a chaotic transient[5]) until the orbit puts x below $-x_*$. After this happens, the orbit rapidly accelerates to large negative values of x.

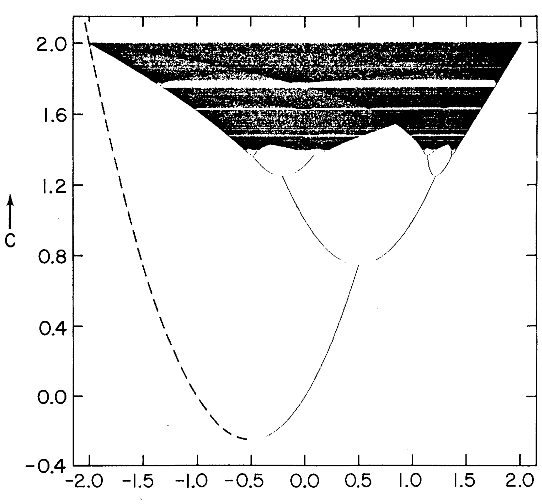

FIG. 1. Bifurcation diagram for the map Eq. (1). The dashed curve is the unstable fixed point. This figure is generated by first preiterating the orbit from an initial condition and then plotting the subsequent orbit in x for a given C, for many different values of C.

One of the points which we wish to convey in this Letter is that such intersections of a chaotic region and a coexisting unstable orbit are prevalent in many circumstances and systems and lead to *discontinuous* qualitative changes in the character of the long-time behavior of the orbits. For example, in the case of the two-dimensional Henon map ($x_{n+1}=1-\alpha x_n^2+y_n$, $y_{n+1}=0.3x_n$), we find, in a certain range of the parameter, α, two strange attractors, each with its own basin of attraction. However, as the parameter is raised, a critical value is reached. At this critical value one of the attractors collides with an unstable (saddle) periodic point on the boundary separating the basins of attraction of the two strange attractors. This collision marks the death of that strange attractor and its basin, and, for values of the parameter immediately above this critical value, that strange attractor is gone. Further discussion of this case will appear in a future publication.[12] In addition, similar crisis-induced deaths of strange attractors and their basins are probably present in several reported ordinary differential-equation examples wherein hysteresis occurs (e.g., in the Lorenz system, as discussed by Kaplan and Yorke,[4] in a model of Josephson junctions given by Huberman and Crutchfield,[2] and in the nonlinear coupled-plasma-wave problem of Russell and Ott[13]).

The example of Fig. 1 concerns a crisis in which the unstable orbit is on the boundary of the basin, and the crisis causes termination of the attractor and its basin (we call this a *boundary crisis*). When the collision occurs within the basin of attraction (we call this an *interior crisis*), a sudden expansion of the attractor almost always occurs. Note that for a boundary crisis the basin of attraction disappears discontinuously, rather than by shrinking continuously to zero (e.g., at the crisis point $C=2$ of Fig. 1, the basin of attraction for the bounded chaotic orbit is $|x|<2$). As an example of an interior crisis, consider Fig. 2. This figure is an enlargement of the bifurcation diagram of Fig. 1 for C between 1.72 and 1.82. This range encompasses the region where stable period-three orbits appear by tangent bifurcation. Also shown in Fig. 2 are dashed curves denoting the *unstable* period-three orbit created at the tangent bifurcation. Note from Fig. 2 that for a range of C less than a certain critical value, C_{*3},

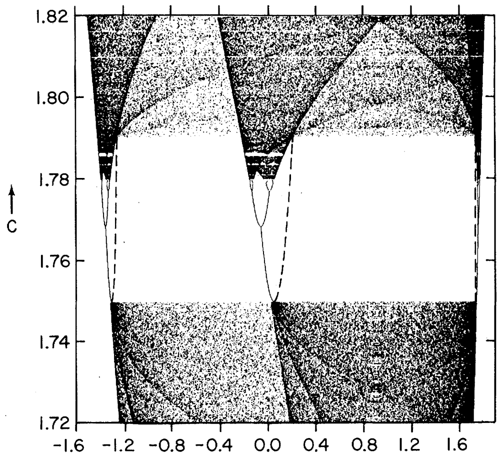

FIG. 2. Blowup of the bifurcation diagram of Fig. 1 in the region of the period-three tangent bifurcation. The dashed curves denote the unstable period-three orbit created at the tangent bifurcation.

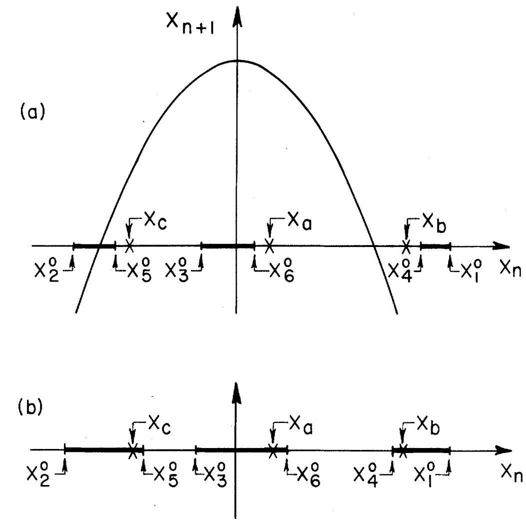

FIG. 3. (a) Schematic illustration of the quadratic map, Eq. (1), for a value of C slightly less than C_{*3}. The three chaotic bands are indicated on the x_n axis with boundary points x_1^0, x_2^0, x_3^0, x_4^0, x_5^0, and x_6^0. Also shown as crosses are the components of the unstable period-three orbit, x_a, x_b, and x_c. (b) Schematic illustration of the x_n axis for C slightly larger than C_{*3}.

chaos occurs in three distinct bands, but that, when C increases past $C_{*3} \simeq 1.79$, the three chaotic regions suddenly widen to form a single band. Furthermore, this coincides precisely with the intersection of the unstable period-three orbit created at the original tangent bifurcation with the chaotic region. We have noted similar crisis-induced widenings associated with the other tangent bifurcations occurring in the chaotic range (i.e., $C_\infty < C \leq 2$, where C_∞ is the accumulation point for period-doubling bifurcations of the original stable fixed point).

Figure 3(a) shows the map (1) for a value of C slightly less than C_{*3}. The three chaotic bands are indicated on the x_n axis as the intervals $[x_3^0, x_6^0]$, $[x_4^0, x_1^0]$, and $[x_2^0, x_5^0]$. Also, the unstable period-three points, x_a, x_b, and x_c, are indicated as crosses. The rightmost boundary of the chaotic region, x_1^0, is clearly the image of $x=0$, since $F(x, C)$ is maximum at $x=0$. Thus, $x_1^0 = F(0, C)$. We denote F composed with itself n times by $F^{(n)}(x, C)$; i.e., $F^{(n)}(x, C) = F(F^{(n-1)}(x, C), C)$, and $F^{(1)}(x, C) \equiv F(x, C)$. Examination of Fig. 3 then shows that $x_n^0 = F^{(n)}(0, C)$, $n = 1$, 2, 3, 4, 5, and 6. Now consider x_4^0. At $C = C_{*3}$, $x_4^0 = x_b$, and hence $x_4^0 = x_7^0$, or

$$F^{(4)}(0, C_{*3}) = F^{(7)}(0, C_{*3}). \tag{2}$$

Equation (2) provides a means for the accurate numerical determination of C_{*3}. We obtain $C_{*3} = 1.790\,327\,492\ldots$.

For C slightly larger than C_{*3}, the unstable orbit x_a, x_b, x_c will lie within the bands $[x_6^0, x_3^0]$, $[x_1^0, x_4^0]$, $[x_2^0, x_5^0]$ [cf. Fig. 3(b)]; x_a will be slightly less than x_6^0, x_b will be slightly greater than x_4^0, and x_c will be slightly less than x_5^0. An orbit started within one of the regions $[x_3^0, x_a]$, $[x_b, x_1^0]$, $[x_2^0, x_c]$ will typically *initially* move about in a chaotic way, cycling between the three regions, as in the case $C < C_{*3}$. After a while, the point will eventually fall within one of the small regions $[x_a, x_6^0]$, $[x_4^0, x_b]$, $[x_c, x_5^0]$. It will then be repelled by the unstable period-three orbit and be pushed into the formerly empty region.

Let f denote the fraction of time which an orbit spends in the formerly empty regions, $[x_5^0, x_3^0]$ and $[x_6^0, x_4^0]$. Consideration of the action of the map leads us to suspect that this fraction will have a functional dependence on $C - C_{*3} \equiv c$ which is approximately of the form $(0 < c \ll 1)$

$$f(c) = c^{1/2} P(\ln c) + k_1 c^{1/2} \ln(k_2/c), \tag{3}$$

where k_1 and k_2 are constants, P is a periodic function, $P(\zeta) = P(\zeta + \alpha)$, and the periodicity α is given by $\alpha \simeq \ln\beta$,

$$\beta = F'(x_a, C_{*3}) F'(x_b, C_{*3}) F'(x_c, C_{*3}),$$

with $F' \equiv dF/dx$. This yields $\alpha \simeq 1.312$. The origin of (3) will be discussed in a future publication.[12] Note that the first term in (3), denoted

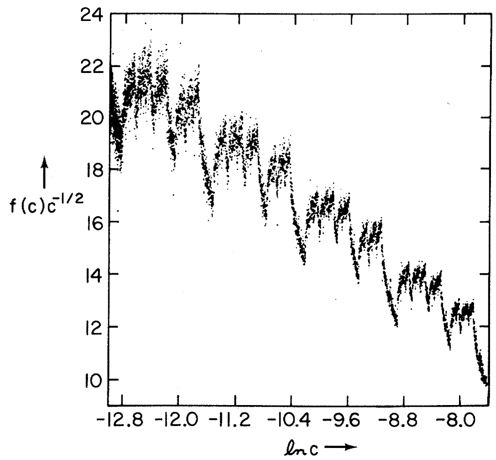

FIG. 4. $f(c)/c^{1/2}$ vs ln$c$. $c = C - C_{*3}$.

$f_1(c)$, is scale invariant, $f_1(\beta c) = \beta^{1/2} f_1(c)$. Figure 4 shows a plot of $c^{-1/2} f(c)$ vs ln$c$ obtained by numerical iteration of Eq. (1). It is seen that the result is in agreement with (3), including the predicted periodicity α.

Now we turn to a consideration of the Lyapunov number of the map at $C = C_{*3}$. For a $C = C_{*3}$ both x_3^0 and x_6^0 map to x_4^0 ($x_3^0 = -x_6^0$). In this case, after three iterations, the middle interval is symmetrically stretched, folded in two, and mapped back onto itself. To the extent that the map F has small curvature in the side intervals $[x_4^0, x_1^0]$, $[x_2^0, x_5^0]$, the map $F^{(3)}$ acting on one of the three intervals is approximately parabolic and stretches and folds the interval in two and then maps it onto itself. Thus, appropriate to this situation, we predict that the Liapunov exponent of $F^{(3)}$ is approximately ln2 and that the Liapunov exponent for F is approximately (ln2)/3. In fact, it can be shown that ln2 is also an exact upper bound for the Liapunov exponent of a map like $F^{(3)}$ (cf. Ref. 12 for a simple proof). Numerical calculation indeed reveals that lnλ, the Liapunov exponent for F, is very close to its upper bound,

$$\ln\lambda \simeq [(\ln 2)/3](1 - 4\times 10^{-4}).$$

Even more precise agreement is found for the case $C = C_{*5}$, corresponding to the crisis point following the tangent bifurcation to a period-five orbit,

$$\ln\lambda \simeq [(\ln 2)/5](1 - 4\times 10^{-6}).$$

In conclusion we have identified two types of crises, *boundary crises* and *interior crises*, and we have illustrated and investigated each within the context of the quadratic one-dimensional map (cf. Figs. 1 and 2). We feel that *boundary crises* are the principal means of sudden destruction of chaotic attractors and their basins in R^n, and that *interior crises* are the principal causes of sudden expansions in the size of chaotic attractors.[14]

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