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Surpassing the Amplifier Limit for Force Detection

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A force-measuring system composed of a macroscopic harmonic oscillator, mechanical-electrical transducer, and linear amplifier is described and the sensitivity is calculated for a "back-action evasion" mode of operation. It is shown that it is possible to surpass the standard amplifier limit with this device, and the specific requirements to do so are given. These results may also apply to a quantum-limited amplifier and thus give the requirements to perform quantum-nondemolition measurements.

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Recent investigations[1] of the quantum mechanical limitations of force-measuring systems have suggested a new type of measurement strategy called back-action evasion. This strategy should increase the sensitivity (for the detection of impulses) by partially isolating the force transducer from the input noise of the first stage of amplification.[2] This technique is potentially important for improving gravity wave detectors, and perhaps for a new class of experimental investigations of quantum measurement theory.

In a recent paper[3] we proposed a specific system which could realize several different modes of force measurement, including the back-action evasion mode proposed by Thorne *et al.*[4] We carried out a complete classical sensitivity analysis of this system and found that the overall system sensitivity for the detection of an impulse could never be better than what might be called the standard amplifier limit.[5] If we express the overall system noise as the number of quanta, n_I, that the noise equivalent impulse would transfer to the unexcited oscillator, and if we express the amplifier noise temperature as a number of quanta, n_A, then the standard amplifier limit is $n_I \geq n_A$.

The purpose of the present paper is to extend our classical analysis to the back-action evasion mode of operation. We show that under certain conditions, one may increase the sensitivity of the system beyond the standard amplifier limit by implementing a back-action evasion strategy. The calculation below includes most of the relevant physical parameters for a realistic system, and so provides a recipe for the experimental observation of back-action evasion. Further, it is possible (but not proven) that our purely classical calculation will contain all the essential features of a fully quantum mechanical calculation. If this is so, simply by setting $n_A = (\ln 2)^{-1}$ we have found the conditions to beat the standard quantum "limit."

The model system under consideration is the same as in our earlier paper[3] and is illustrated in Fig. 1. It is an accelerometer with electrodes placed to make a "balanced" three-plate capacitor. This is part of a bridge circuit which has a resonant readout, the current through which is sensed by a linear amplifier; we chose a SQUID for concreteness, but the equivalent circuit may represent any linear amplifier with a small input impedance. To operate this in the back-action

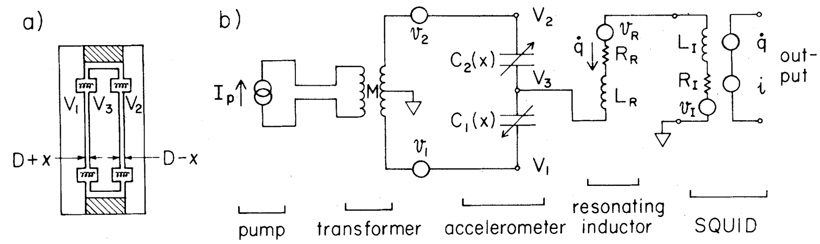

FIG. 1. (a) A schematic diagram of an accelerometer. An impulse applied to the outer plates causes the relative coordinate x to oscillate. (b) The electrical schematic of the bridge circuit used to sense the motion of x. Noise generators are labeled with italic letters.

evasion mode, we must choose the bridge excitation, or pump voltage, to be

$$M\dot{I}_p = \tfrac{1}{2}V_0[\cos(\omega_2+\omega_1)t + \cos(\omega_2-\omega_1)t] \equiv V_p(t),$$

where ω_1 and ω_2 are the angular frequencies of free oscillation for the mechanical and readout oscillators, respectively. This can be interpreted as the coherent superposition of a parametric amplifier pump and a parametric up-convertor pump.

From the classical viewpoint, the measurement of a "signal" force $F(t)$ applied to the outer plates is only limited by the noise sources which couple in various ways to the system and cause fluctuations at the amplifier output. We consider four noise sources: (1) the Langevin force $f(t)$, which is determined by the mechanical loss coefficient Q_1^{-1} and the temperature T; (2) the Johnson noise voltage $v_J(t) \equiv \frac{1}{2}(v_1+v_2)+v_R$, which is determined by the electrical loss coefficient Q_2^{-1} and T; (3) the input noise $v_I(t)$ of the amplifier; and (4) the additive noise $i(t)$ of the amplifier, referred to its input. Pump noise has no effect if the capacitances are perfectly balanced and we defer a full discussion of this problem to a future publication.[6] In the calculation below, we find that the last three noise sources enter the result only as the combination we call the total electrical noise number $n_E \equiv [S_v(\omega_2)S_i(\omega_2)]^{1/2}/\hbar\omega_2$, where S_i is the double-sided spectral density of the random variable i, etc., and $v \equiv v_J + v_I$. When $S_{v_J} \ll S_{v_I}$, we have $n_E \to n_A$, the amplifier noise number. The first noise source enters the result only in the value of the dimensionless parameter $\alpha \equiv (k_B T/\hbar\omega_1 Q_1)/n_E$.

We have found that the equations of motion for the relative displacement $x(t)$, and the net charge on the center capacitor plate $q(t)$, are most easily solved in the back-action evasion mode by transforming them into equations for the (real-valued) components X_1, X_2, Q_1, and Q_2 of the "complex amplitudes" of the two oscillators, also called the in-phase and quadrature amplitudes. They are defined in the classical case by

$$X_1(t)+jX_2(t) \equiv [x(t)+j\omega_1^{-1}\dot{x}(t)]\exp(+j\omega_1 t), \tag{1}$$

$$Q_1(t)+jQ_2(t) \equiv [q(t)+j\omega_2^{-1}\dot{q}(t)]\exp(+j\omega_2 t), \tag{2}$$

where $j \equiv \sqrt{-1}$. The equations of motion are approximately

$$\left(\frac{d}{dt}+\frac{1}{2\tau_1}\right)X_1 = \frac{V_0}{D}\frac{1}{4\mu\omega_1}Q_1\sin(2\omega_1 t) - \frac{F(t)}{m\omega_1}\sin(\omega_1 t) - \frac{f(t)}{\mu\omega_1}\sin(\omega_1 t), \tag{3}$$

$$\left(\frac{d}{dt}+\frac{1}{2\tau_1}\right)X_2 = \frac{-V_0}{D}\frac{1}{4\mu\omega_1}Q_1[1+\cos(2\omega_1 t)] + \frac{F(t)}{m\omega_1}\cos(\omega_1 t) + \frac{f(t)}{\mu\omega_1}\cos(\omega_1 t), \tag{4}$$

$$\left(\frac{d}{dt}+\frac{1}{2\tau_2}\right)Q_1 = \frac{-v(t)}{\omega_2 L}\sin(\omega_2 t), \tag{5}$$

$$\left(\frac{d}{dt}+\frac{1}{2\tau_2}\right)Q_2 = \frac{-V_0}{D}\frac{1}{4L\omega_2}\{X_1[1+\cos(2\omega_1 t)]+X_2\sin(2\omega_1 t)\} + \frac{v(t)}{\omega_2 L}\cos(\omega_2 t), \tag{6}$$

where m is the mass of the outer plates, μ is the reduced mass, D is the mean capacitor gap, $L \equiv L_R + L_I$, $R \equiv R_R + R_I$, $\omega_2 \equiv (2C_0 L)^{-1/2}$, $\tau_1 \equiv Q_1/\omega_1$, $\tau_2 \equiv Q_2/\omega_2$, and C_0 is the mean plate capacitance of one side.

We have dropped all the coupling terms which oscillate at high frequencies ($\sim 2\omega_2$), because their effect averages nearly to zero over the time scale of interest, but we have not dropped those terms which oscillate at $2\omega_1$; this is appropriate when $\omega_2 \gg \omega_1$, and the coupling is strong enough to allow the optimum overall system bandwidth to approach ω_1. All signal and noise terms have been kept.

To calculate the weakest detectable impulsive signal $F(t)=p\delta(t-t_I)$, we then followed the procedure outlined earlier[3]; this procedure has the disadvantage of obscuring the qualitative behavior of the system but has the advantage of making precise the notion of overall system sensitivity. The system of equations was solved by Fourier transformation and making successive substitutions, beginning with Eq. (5). The main difference was that the output of the amplifier $[\dot{q}(t)+i(t)]$ was demodulated with $\cos(\omega_2 t)$ and filtered to find the apparent value of $\tilde{Q}_2$, rather than the apparent $\tilde{x}$. The noise and signal content of the apparent $\tilde{Q}_2$ was calculated, and then the optimally filtered noise equivalent impulse $\langle p_n{}^2\rangle^{1/2}$ was found.[7] The result is expressed as the overall system noise number $n_I \equiv \langle p_n{}^2\rangle/(2m^2\hbar\omega_1/\mu)$. Making a number of approximations[8] we find the result

$$\frac{n_I}{n_E} \simeq \pi[\sin^{-2}(\omega_1 t_I)]\left[\int_{-\infty}^{+\infty} dy\left(\alpha+\frac{3}{64}\beta\gamma+\frac{8\gamma}{\beta}y^2+\frac{8}{\beta\gamma}y^4\right)^{-1}\right]^{-1}, \tag{7}$$

where we define the coupling coefficient $\beta \equiv (V_0/D)^2(\omega_2/\omega_1)(C_0/\mu\omega_1{}^2)$, the impedance ratio $\gamma \equiv (\omega_2/\omega_1)(\omega_2 C_0)[S_v(\omega_2)/S_i(\omega_2)]^{1/2}$, and the dimensionless frequency $y \equiv \omega/\omega_1$.

This integral may be evaluated analytically, with the result that at the optimum arrival times,

$$\frac{n_I}{n_E} \simeq \left\{\frac{8\alpha'}{\beta}\left[\gamma+\left(\frac{\alpha'\beta}{2\gamma}\right)^{1/2}\right]\right\}^{1/2}, \tag{8}$$

where $\alpha' \equiv \alpha + 3\beta\gamma/64$.

Figure 2 is a plot of the final result calculated from Eq. (8), with γ optimized for each value of α and β. We find that the back-action evasion mode offers improvement over the resonant bridge for all α and β on the plot. Better yet, we see that if $\alpha \ll 1$ and $\beta \sim 1$, it is possible to exceed the standard amplifier limit ($n_I \geq n_E \simeq n_A$) by a significant amount.

As an example, take the parameters considered in our earlier paper, where we found that $n_I/n_E \simeq 1$ for $\alpha = 10^{-3}$ and $\beta = 0.2$ (and the optimum $\gamma = 2 \times 10^{-1}$). For the same α and β (and the new optimum $\gamma = 1.6 \times 10^{-2}$), the back-action evasion mode improves this to $n_I/n_E \simeq 0.07$.

It has been pointed out by Caves[9] that our final expression [Eq. (8)] reduces, in a certain limit, to one obtained in previous analyses. When the Brownian noise is much smaller than the back reaction (i.e., $\alpha \ll 3\beta\gamma/64$) and when the amplifier-transducer combination is quantum limited (i.e., $n_E = 1$) then $n_I \gtrsim \gamma\sqrt{\tfrac{3}{8}}$ and $\gamma = (\omega_2/\omega_1)(\omega_2 C_0) \times S_v(\omega_2)/\hbar\omega_2$; thus we see that S_v should be as small as possible. If we can further reduce T to the point where S_v becomes dominated by the zero-point fluctuations in the circuit (i.e., $S_v = \hbar\omega_2 R$), then we obtain $n_I \gtrsim 0.61(\omega_1\tau_2)^{-1}$, which is essentially Eq. (1.21a) of Ref. 1 and Eq. (33) of Ref. 10.

The major price for this improved performance is that the response of the system is phase sensitive, similar to a lock-in amplifier; note the factor $\sin^{-2}(\omega_1 t_I)$ in Eq. (7). There is an important difference, though; for a back-action evasion scheme, the phase sensitivity precedes the first

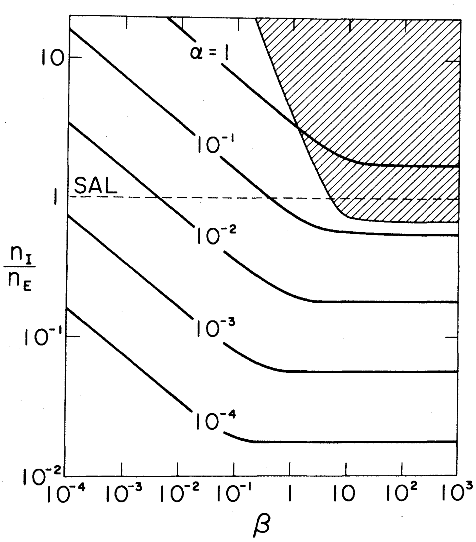

FIG. 2. A plot of the overall system noise number n_I vs the electromechanical coupling coefficient β for a series of different values of α (α is proportional to the ratio of mechanical to electrical noise power). The impedance matching coefficient γ has been optimized. The broken line represents the standard amplifier limit (SAL) which is impossible to exceed with non–back-action-evasion measurement techniques. The shaded region represents those combinations of dimensionless parameters for which the averaging time of the overall system is less than one cycle of the oscillator and the approximations used in the calculation probably break down.

stage of amplification.[1]

Why does this measurement strategy work? The answer may be found by considering that part of the noise, called the back action, which is caused by electrical noise perturbing the mechanical oscillator. The electrical noise is the fluctuation of Q_1 and Q_2 caused by the noise generator v. By inspecting the right-hand sides of Eqs. (3) and (4), we find that the electrical noise has much less effect on X_1 than on X_2, because all the terms with an oscillating coefficient *almost* average to zero. Similarly the response of Q_2 to X_2 is nearly averaged away; so the transducer serves a dual purpose, first to partially isolate X_1 from the back reaction of the amplifier and second to partially isolate the "interesting" Q_2 phase of the transducer output from the back-reaction–contaminated X_2 phase of the mechanical oscillator.

Because X_1 is not completely decoupled from Q_1, and because Q_2 is not completely decoupled from X_2, there is a limit to the improvement available to this type of back-action evasion. The residual couplings cause the curves of constant α (Fig. 2) to become flat for large values of β. Then the only way to effect further improvement is to reduce the size of the noise sources, i.e., reduce n_E or T/Q_1.

This paper describes, in purely classical terms, a method to circumvent the limitations imposed by a classical noisy amplifier. Whether or not this has any bearing on the performance of quantum-nondemolition measurements can only be decided by an analysis of a full quantum version of our system including the amplifier. This poses a number of new questions intimately connected with quantum measurement theory which we do not propose to solve here. There are assertions in the literature[1,10] that an analysis such as ours will give the same limit as a full quantum version, provided we let $n_A \to (\ln 2)^{-1}$. If this is correct, our result predicts the sensitivity of a physically realizable device which will be able to beat the quantum "limit."

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[1]C. M. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmermann, Rev. Mod. Phys. 52, 341 (1980).

[2]The problem of "back reaction" in this context was first seriously considered by Braginsky. See V. B. Braginsky and A. B. Manukin, in *Measurement of Weak Forces in Physics Experiments*, edited by D. H. Douglass (Univ. of Chicago Press, Chicago, 1977).

[3]W. W. Johnson and M. Bocko, Phys. Rev. Lett. 47, 1184 (1981).

[4]K. S. Thorne, C. M. Caves, V. D. Sandberg, M. Zimmermann, and R. W. P. Drever, in *Sources of Gravitational Radiation*, edited by Larry Smarr (Cambridge Univ. Press, London, 1979), p. 49.

[5]This is essentially the result first found by R. P. Giffard, Phys. Rev. D 14, 2478 (1976).

[6]The pump noise does not contribute if

$$(x_0/D)^2 S_\varphi(\omega_1) < (2/V_0)^2 S_v(\omega_2) = 4(\gamma\hbar n_E/\beta)(\mu\omega_1{}^2 D^2)^{-1},$$

where x_0/D is the fractional imbalance of C_1 and C_2, and where $S_\varphi(\omega_1)$ is the phase-noise spectral density at a frequency offset ω_1. There is the same requirement on the amplitude noise. Of course, the pump-noise requirement is more severe [by the factor $(D/x_0)^2$] for single-capacitor motion transducers. The practical limits on balancing and pump noise require an extensive discussion which we leave for a future publication.

[7]Athanasios Papoulis, *Signal Analysis* (McGraw-Hill, New York, 1977), especially Chap. 10.

[8]The inclusion of all the coefficients which oscillate at $2\omega_1$ makes the algebra rather lengthy, but by use of $\omega_1\tau_1, \omega_1\tau_2 \gg 1$ we find that many of the final terms become relatively small or have a simple polynomial approximation. All of the structure in the integrand near $y=2$ can be dropped throughout the unshaded portion of Fig. 2, because it is dominated by other terms. We optimized γ by numerical variation of Eq. (7).

[9]Carlton M. Caves, personal communication.

[10]V. B. Braginsky, Y. I. Vorontsov, and K. S. Thorne, Science 209, 547 (1980).