# Drift-Wave Turbulence from a Soliton Gas

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A theory of drift-wave turbulence as a low-density gas of drift-wave solitons is developed. In the ideal-gas approximation to the many-soliton system the dynamical form factor $S(k,\omega)$ is computed and shown to peak at $\omega > kv_d$ with a substantial width $\Delta\omega$ for a given azimuthal wave number.

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In this Letter we propose, as an alternative theoretical framework for the interpretation of the plasma fluctuation measurements,[1-3] the concept of an (nearly) ideal gas of drift-wave solitons. Similar ideas have recently been studied in the context of condensed matter physics,[4] where solitons are shown to contribute to the free energy of the nonlinear lattice.[5]

The turbulence theory presented here, although interesting in itself as an alternative representation of chaotic fields in terms of coherent nonlinear objects, was principally motivated by the apparent inability of renormalized turbulence theories to account for the frequency spectrum measured by electromagnetic scattering experiments in tokamak plasmas. With a given scattering geometry determining the fluctuation wave number $k$, the observed frequency dependence of the dynamical form factor $S(k,\omega)$ for the electron density fluctuations $\langle|\delta n_e(k,\omega)|^2\rangle$ is peaked at frequency $\omega_p(k)$ greater than the linear mode frequency $\omega^l(k)$ and has a width $\Delta\omega(k)$ that is greater than or comparable to $\omega^l(k)$. An explanation for the location of the peak frequency $\omega_p(k)$ can be given in terms of a Doppler shift[6] of the entire spectrum from an ambipolar radial electric field; however, the width $\Delta\omega(k)$ of the observed spectrum exceeds that given by renormalized turbulence theory and remains unexplained.

The broad frequency spectrum may lie outside the scope of renormalized theories wherein the truncation of high-order correlations, based for example on the assumption[7] of "maximal randomness," leads to $\Delta\omega(k) \propto \langle|\delta n_e(k)|^2\rangle \lesssim \omega^l(k)$ for the fluctuation levels observed. An example of the application of renormalized turbulence theory is given by the formulas of Horton[8] for

$$I(k)=\langle|e\Phi(k)/T_e|^2\rangle=\langle|\delta n_e(k)|^2\rangle/n_e^{\,2}$$

and $\Delta\omega(k)=\nu_k$ in a simple two-dimensional hydrodynamic model for drift-wave turbulence. The soliton gas theory of drift-wave turbulence presented here overcomes this difficulty in a natural manner by using highly correlated objects as a nonlinear basis for representing the drift-wave fields. After introducing a simple one-dimensional nonlinear model[9] for the drift-wave problem, we show that a typical root mean square level $\varphi_0=\langle e^2\Phi^2/T_e^{\,2}\rangle^{1/2}$ of the drift-wave field system may contain a large number, $N_s$, of solitons. In the ideal-gas limit we show that the width of the frequency spectrum $\Delta\omega(k)$ is proportional to $\varphi_0$ rather than $\varphi_0^{\,2}$.

The pressureless ion limit of the two-fluid description of drift waves is based on the ion continuity equation $\partial_t n_i+\nabla\cdot(n\vec{v}_i)=0$ with $\vec{v}_i$ given by the $\vec{E}\times\vec{B}$ drift $\vec{v}_E=c\hat{b}\times\nabla\Phi(\vec{r},t)/B$ and the nonlinear polarization drift $\vec{v}_p=(c^2m_i/e_iB^2)(d\vec{E}_\perp/dt)$. The electron density evolves through the quasistatic equilibrium given by

$$n_e(\vec{r},t)=n(x)\exp[e\Phi(\vec{r},t)/T_e(x)],$$

where the density and electron temperature $T_e(x)$ are nonuniform in the radial direction $x$ across the constant magnetic field $B\hat{z}$. We define the scale lengths for the density and temperature variations by $r_n^{-1}=-\partial_x \ln n(x)$ and $\eta_e=\partial_x \ln T_e(x)/\partial_x \ln n(x)$, and the scale length for wave dispersion by ion inertial length $\rho=c(m_iT_e)^{1/2}/eB$. Invoking quasineutrality determines the general nonlinear wave equation[8-10] for $\Phi(\vec{r},t)$. Here we consider the small-$k_\perp\rho$, one-dimensional limit of the general equation. For $(k_\perp\rho)^2<1$ and $(k_x\rho)\times(k_\perp\rho)^2 \ll \eta_e(\rho/r_n)$, the equation first given by Petviashvili[9] reduces[11] to the Korteweg–de Vries (KdV) equation

$$\partial_t\varphi+\partial_y\varphi+\partial_y^{\,3}\varphi-\varphi\,\partial_y\varphi=0. \tag{1}$$

In Eq. (1) $\varphi(y,t)=\eta_e e\Phi(\vec{r},t)/T_e$; $y$ is measured in units of $\rho$, and time $t$ in units of $\rho/v_d$ where the electron diamagnetic drift velocity is $v_d=\rho c_s/r_n=cT_e/eBr_n$. The radial structure of the solutions of Eq. (1) is given by a linear eigenmode problem involving $v_d(x)$ and $T_e(x)$.[12] The eigenmode width is roughly $\Delta x = (\rho r_n)^{1/2}$. The system is periodic in the $y$ direction with length $L$ which is typically $2\pi r$ where $r$ is the radius of the confinement device at the maximum of $v_d$. The energy of the drift-wave field is given by $E = \frac{1}{2}\int \varphi^2 dy$ measured in units of $nT_e \Delta x L_c / \eta_e^2$.

We propose that a turbulent state described by Eq. (1) will consist of a broad wave-number spectrum, $I(k)$, of small-amplitude modes with $\omega^l(k) < kv_d$ together with an ensemble of solitons $\varphi_s(y - ut)$ with $u > v_d$. The soliton "dispersion relation" $\omega = ku$ determines the frequencies at which the solitons contribute to the spectrum. As a first approximation we ignore the small-amplitude component, supposing that its spectrum $(I(k), \omega^l(k))$ can be added to that derived here for the drift-wave solitons.

The soliton component to the drift-wave field is

$$\varphi(y,t) = \sum_{n=1}^{N_S} \varphi_s(y,t;y_n,u_n), \tag{2}$$

where

$$\varphi_x(y,t;y_0,u) = -3(u-1)\operatorname{sech}^2[\tfrac{1}{2}(u-1)^{1/2}(y - y_0 - ut)]. \tag{3}$$

In writing Eq. (2) we ignore the overlap of the solitons, assuming that their density is small. Furthermore, it is well known that soliton-soliton collisions are completely elastic, giving only a phase shift of the soliton positions.

The inverse scattering transformation allows the determination of the number density $n(u)$ of solitons emitted by any particular initial state. For moderate-amplitude initial states, $\varphi \gg (\rho/r_n)^2$, the total number of solitons produced is large and a WKB approximation of the inverse problem can be used to obtain[13]

$$n_u[\varphi] = \frac{\sqrt{3}}{4\pi} \int_{\varphi < -A/2} \frac{dy}{[-2\varphi(y) - A]^{1/2}}, \quad A = 3(u-1), \tag{4}$$

where $A$ is the soliton amplitude. This formula is strictly valid only for initial conditions with $\varphi(y) < 0$ for all $y$ (otherwise significant nonsoliton excitations affect the distribution), but will be used here as a first approximation.

We introduce a statistical description through an average over initial configurations. A more complete theory of drift waves would include various linear growth and damping terms which would cause an infinitesimal disturbance to grow and evolve according to weak-turbulence theory. Eventually, the system will saturate at some randomly phased spectrum $\langle|\varphi_k|^2\rangle$ where wave numbers up to $k_\rho \sim 1$ are excited. At this point, if the amplitude is large enough, strong correlations will develop as a result of the nonlinearity as implied by Eq. (1). Thus we take as the initial state for Eq. (1) a random-phased field $\varphi$ with some given mean square amplitude $\langle|\varphi|^2\rangle = \varphi_0^2$.

The distribution of solitons emerging from this state may be obtained by averaging Eq. (4) over the random phases of the initial configuration. Since the width of the spectrum in $k$ space is much larger than that of the typical soliton wave number allowed by the approximate Eq. (1), we can take the initial spectrum to be white noise. The average over Eq. (4) is then a Gaussian functional integral which can be done by discretization:

$$f_s(u) = \frac{1}{Z} \prod_{i=1}^{n} \int_{-\infty}^{\infty} d\varphi_i \exp\left[\frac{-\varphi_i^2}{2\varphi_0^2}\right] n_u[\varphi], \tag{5}$$

where $\varphi_i$ represents $\varphi(x_i)$ and $x_i = (i/n)L$. Here we have defined the mean soliton distribution function $f_s(u)$ such that

$$\int_{v_d}^{\infty} f_s(u) du = N_s,$$

where $N_s$ is the mean number of solitons. Upon conversion of the integral in Eq. (4) for $n_u$ to a sum, the integral in Eq. (5) is easily done:

$$f_s(u) = \frac{\sqrt{3}}{8\pi}\left(\frac{L}{\varphi_0^{1/2}}\right) \exp\left[-\frac{A^2}{16\varphi_0^2}\right] D_{-1/2}\left(\frac{A}{2\varphi_0}\right), \tag{6}$$

where $D_{-1/2}$ is a parabolic cylinder function.[14] The total number of solitons is then

$$N_s = \int_{v_d}^{\infty} du\, f_s(u) = \alpha L \varphi_0^{1/2}, \quad \alpha = \Gamma(\tfrac{3}{4})/(12\sqrt{2}\pi^3)^{1/2}. \tag{7}$$

The spectral density is the Fourier transform of the two-point correlation function, $\langle\varphi(x+\xi, t+\tau)\varphi(x,t)\rangle$. Use of the soliton field of Eq. (2) with Eq. (3) gives

$$S(k,\omega) = \frac{288}{L} k^2 f_s\left(\frac{\omega}{k}\right) \operatorname{csch}^2\left[\pi k \left(\frac{kv_d}{\omega - kv_d}\right)^{1/2}\right], \tag{8}$$

where, since $N_s \gg 1$, the sum over solitons in Eq. (2) has been converted to an integral with the distribution function as weight.

Several features of the spectrum are evident directly from Eq. (6) independent of the explicit form of $f_s(u)$. First, $S(k,\omega) = 0$ for $\omega < kv_d$ because all solitons have $u > v_d$. The spectrum is exponentially small near $\omega \sim kv_d$ and peaks at $\omega \sim kv_d[1 + O(\varphi_0)]$.

Approximation of the parabolic cylinder function in Eq. (6) and substitution into Eq. (8) yields

$$S(k,\omega)\propto\begin{cases} k^2\exp\left[\dfrac{-A^2}{16\varphi_0^2}\right]\operatorname{csch}^2\left[\dfrac{\sqrt{3}\pi k}{A^{1/2}}\right], & A\ll\varphi_0,\\ k^2A^{-1/2}\exp\left[\dfrac{-A^2}{8\varphi_0^2}\right]\operatorname{csch}^2\left[\dfrac{\sqrt{3}\pi k}{A^{1/2}}\right], & A\gg\varphi_0,\end{cases}$$

where $A=3(\omega/kv_d-1)$. The peak of the spectrum occurs at $\omega_p\simeq kv_d(1+0.7\varphi_0)$ for $(\pi k)^2\ll\varphi_0$ and at $\omega_p\simeq kv_d[1+0.7(\pi k\varphi_0^2)^{2/5}]$ for $(\pi k)^2\gg\varphi_0$. The roughly linear increase of $\omega_p$ with $k$ agrees with the experiments in the region $k\rho<1$. The spectral width also increases linearly with $k$ and $\varphi_0$.

Several important effects which have been omitted from our model will contribute to the width of the spectrum seen in an experiment. Some fraction of the fluctuation energy will be contained in nearly linear modes. This contribution to the spectrum will peak at $\omega=\omega^l(k)<kv_d$. Another spectral contribution will arise from higher-order dispersion and other perturbations which should be added to Eq. (1). The resulting equation (e.g., Petviashvili[9]) will be no longer completely integrable and soliton collisions will be no longer elastic. A collision between two solitons with speeds $u_1$ and $u_2$ would excite frequencies $\omega=k_1u_1-k_2u_2$ with $k=k_1-k_2$. These driven modes will contribute to $S(k,\omega)$ in the region $\omega<kv_d$. Finally, we will show in a future work that negative-velocity solitons are excited for larger $k\rho$ and these yield a spectral peak at $\omega\lesssim 0$.

We do not regard the limitation of the results of this paper to the one-dimensional case as fundamental. The Petviashvili equation possesses two-dimensional solitary waves[9] qualitatively similar to Eq. (3), and therefore the frequency spectrum will also resemble ours qualitatively.

In conclusion, we suggest that from both a theoretical and an experimental point of view, a full understanding of drift-wave turbulence may require a theory of both the continuum component $\omega_{\vec{k}}^{\,l}$, $I(\vec{k},t)$ (as in conventional turbulence theory) and the soliton component $\vec{k}\cdot\vec{u}$, $f_s(\vec{u},t)$ of the turbulent plasma.

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