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Resolution of the Einstein-Podolsky-Rosen and Bell Paradoxes

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A model of spin-$\frac{1}{2}$ statistics that explains the observed frequencies on the basis of the validity of the principle of locality is proposed. The model is based on the observation that certain density conditions on the unit sphere correspond with the observed frequencies while the resulting expectation values violate Bell's inequality.

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Bell[1] has observed that no hidden-variable theory satisfying a principle of locality can reproduce the quantum statistics of electron pairs in the singlet spin state. Bell's argument was simplified by Wigner[2] and put in its most general testable form by Clauser and Horne.[3] Various experiments[4] designed to test the locality principle have shown the observed frequencies to conform with quantum mechanics (i.e., to violate Bell's inequality). As a result, a wide variety of so-called "objective local theories" were ruled out. There is, however, a logical possibility that has escaped attention. It might be the case that there is nothing wrong with the locality principle, and the violation of Bell's inequality indicates rather a limitation in the mathematical theory of probability. In other words, we can conceive of mathematical situations where a natural concept of probability emerges which is not captured by the usual (Kolmogorov) axioms of probability theory. What I have in mind is not a radical extension of probability (such as introducing negative or complex probability values) but rather a conservative extension as will be clarified below. The purpose of this article is to turn this logical possibility into a physical model that explains the observed frequencies on the basis of the validity of the locality principle. A more detailed account that includes complete proofs and generalizations to other spin (angular momentum) states, as well as some predictions, will be published shortly.

Let $S^{(2)}$ be the (surface of a) unit sphere in three-dimensional Euclidean space: $S^{(2)}=\{x \in E^{(3)} \mid |x|=1\}$. Define a spin function as any function, $s: S^{(2)} \to \{-\frac{1}{2},\frac{1}{2}\}$, which satisfies $s(-x) = -s(x)$. The purpose of the first part of this paper is to develop some *mathematical* constraints on spin functions which are treated purely hypothetically at this stage, without introducing any physical assumptions.

Every spin function divides the sphere into two halves, $\{x \mid s(x)=\frac{1}{2}\}$ and $\{x \mid s(x)=-\frac{1}{2}\}$. In the general case, we suppose that we do not know the exact values of a spin function s but that we have some information, statistical in nature, about the way the set $\{x \mid s(x)=\frac{1}{2}\}$ is distributed over the sphere. I shall deal with a particular type of such information. Let $y \in S^{(2)}$ and $0<\theta<\pi$. Denote by $c(y,\theta)$ the set of all unit vectors that form an angle θ with y, that is $c(y,\theta)=\{x \mid \hat{x}\cdot\hat{y}=\cos\theta\}$. $c(y,\theta)$ is a circle on the sphere with radius $\sin\theta$ and center on the vector y (or $-y$). Let m_θ be the Lebesgue measure *on the circle* $c(y,\theta)$, so that $m_\theta[c(y,\theta)]=2\pi\sin\theta$. Let s be a spin function. Then if the set $\{x \mid s(x)=\frac{1}{2}\} \cap c(y,\theta)$ is m_θ measurable, the expression $(2\pi\sin\theta)^{-2} m_\theta[\{x \mid s(x)=\frac{1}{2}\}$

$\cap\, c(y,\theta)]$ is the (average) density of $\{x \mid s(x)=\frac{1}{2}\}$ in $c(y,\theta)$. We have the following:

Existence theorem.—There exists a spin function s such that for all $y \in S^{(2)}$ and all $0<\theta<\pi$ the set $\{x \mid s(x)=\frac{1}{2}\} \cap c(y,\theta)$ is m_θ measurable and

$$\frac{m_\theta[\{x \mid s(x)=\frac{1}{2}\} \cap c(y,\theta)]}{2\pi \sin\theta} = \begin{cases} \cos^2(\frac{1}{2}\theta) & \text{if } s(y)=\frac{1}{2}, \\ \sin^2(\frac{1}{2}\theta) & \text{if } s(y)=-\frac{1}{2}. \end{cases} \tag{1}$$

The complete proof of the theorem will be published separately. The existence theorem belongs to a family of "strange" or seemingly "paradoxical" results that one can prove in set theory. The proof involves transfinite induction on circles and is based on two observations. Firstly, that the intersection of two nonidentical circles contains at most two points and, secondly, that any subset of $c(y,\theta)$ whose coordinality is strictly less than the continuum is m_θ measurable and has m_θ measure zero. To ensure that the second premise is true, we have to assume the validity of the continuum hypothesis, or at least the validity of the (strictly) weaker Martin's axiom.[5] It is important to note that there exists no analytic expression or algorithm by which one can calculate the values of a spin function that satisfy Eq. (1) for the different directions. In fact, the set $\{x \mid s(x)=\frac{1}{2}\}$ turns out to be nonmeasurable in terms of the Lebesgue measure *on the sphere* and the existence theorem may turn out to be independent of the usual axioms of set theory. The proof of the theorem actually establishes the existence of infinitely many spin functions that satisfy (1).

Let $\mathfrak{F}$ be the family of spin functions with this property and let O_3 be the group of orthogonal transformations in $E^{(3)}$. If $s \in \mathfrak{F}$ and $\alpha \in O_3$ then the spin function $s \circ \alpha$ defined by $s \circ \alpha(x) = s[\alpha(x)]$ is also in $\mathfrak{F}$. This follows from the fact that the density condition applies uniformly to all circles of a given radius. Thus $\mathfrak{F}$ is O_3 invariant. There are subsets of $\mathfrak{F}$ which are invariant as well. With $s \in \mathfrak{F}$ fixed, the set $\{s \circ \alpha \mid \alpha \in O_3\}$ is trivially invariant and is in fact infinite. This can easily be proved if we observe that the group of "real rotations" (without reflections) in $E^{(3)}$ is simple (i.e., has no nontrivial invariant subgroups).

Suppose we know that a given spin function s satisfies condition (1), but we do not know the values of s except at one point, $s(y)=\frac{1}{2}$, say. Then it is reasonable to interpret the expression $(2\pi \sin\theta)^{-1} m_\theta[\{x \mid s(x)=\frac{1}{2}\} \cap c(y,\theta)]$ $[=\cos^2(\frac{1}{2}\theta)$ in this case$]$ as the probability that $s(x)=\frac{1}{2}$ for $x \in c(y,\theta)$. Thus $\cos^2(\frac{1}{2}\theta)$ is the *conditional probability* of $s(x)=\frac{1}{2}$ given $s(y)=\frac{1}{2}$ for a single $s \in \mathfrak{F}$ and all x, y which satisfy $\hat{x}\cdot\hat{y}=\cos\theta$.

Note that s is a *single* spin function which has definite values everywhere on the sphere—our use of probabilities reflects our ignorance of these values.

I have interpreted formula (1) as an expression for conditional *probabilities*. A natural question to ask is whether we can find a probability space from which we get the values of (1) by conditionalization. In other words we are looking for a probability space such that for all $y \in S^{(2)}$ the event "spin up in the y direction" is defined and has probability $\frac{1}{2}$. Also we want that for all x and y the probability of the joint event "spin up in the x direction and spin up in the y direction" will be $\frac{1}{2}\cos^2(\frac{1}{2}\theta)$, where θ is the angle between x and y. With use of Bell's inequality one can prove that no such probability space exists.[6] [Roughly speaking the values $\frac{1}{2}\cos^2(\frac{1}{2}\theta)$ are incompatible with the additivity axiom for probability.] My way out of this problem is to interpret $\cos^2(\frac{1}{2}\theta)$ as the conditional expectation for "spin up" on a circle, given that the spin is up in the center of the circle. From this perspective Bell's theorem shows that one cannot "collect" all these conditions and represent them uniformly on a single probability space. From a mathematical standpoint it seems therefore natural to extend the concept of probability to include conditions of this kind. The physical significance of this extension is that it corresponds with relative frequencies of observable events.

Let us adopt the extreme realist position and maintain that every electron at each given moment has a definite spin in all directions. That is, we associate a spin function s with every electron at every given moment (different electrons may have different spin functions). We do not know the values of s for any given electron but assume that *all* electron spin functions share a common property, namely, they all belong to a family $\mathfrak{F}_0$ of the form $\{s_0 \circ \alpha \mid \alpha \in O_3\}$, where s_0 is some fixed (yet unknown) spin function that satisfies the density formula (1). As the result of an interaction an electron spin function may be transformed to another spin function. I assume, however, that the transformation is always of the form $s \to s \circ \alpha$, where α is a rotation or reflection of space. The orthogonal transformation α depends on the type of interaction and on the initial conditions—the precise nature of this dependence is unknown. (A possible speculation: It may be somehow related to the so-called Wigner rotations.)[7]

The present model proposes that all electron spin functions satisfy the density formula (1). In other words, the density values (or the conditional probabilities) are *invariant*. I also assume that any spin function $s \in \mathfrak{F}_0$ has the same chance of being realized as any other function of $\mathfrak{F}_0$; this may be regarded as a reflection of the isotropy of space.

Suppose that we pass a beam of electrons through a Stern-Gerlach apparatus in a given y direction. As a result of this measurement each one of the electron spin functions will undergo an orthogonal transformation. I assume that the measurement does not change the spin value in the y direction itself, and thus the orthogonal transformations leave y invariant in this case. If we now take the subbeam of the original beam which is polarized up in the y direction and pass it through a second Stern-Gerlach apparatus in a different x direction, the relative frequency of the event "spin up in the x direction" in this subbeam will be approximately $\cos^2(\frac{1}{2}\theta)$, where θ is the angle between x and y. The reason for this is as follows: Let $s_1, \ldots, s_n$ be the spin functions of the electrons in the subbeam. All these functions satisfy the density condition (1), and also $s_j(y) = \frac{1}{2}$ for $j = 1, \ldots, n$; therefore if $A_j = \{x \mid s_j(x) = \frac{1}{2}\} \cap c(y, \theta)$, then $m_\theta(A_j)/2\pi \sin\theta = \cos^2(\frac{1}{2}\theta)$ for $j = 1, \ldots, n$. I have also assumed that every function of $\mathfrak{F}_0$ has the same chance of being realized and thus, apart from the fact that the beam is polarized up in the y direction, the sample is random with respect to its spin functions. This means that the sets A_j are mutually independent in $c(y, \theta)$. It follows from the (strong) law of large numbers that

$$\frac{1}{n}\sum_{j=1}^{n} \chi_{A_j}(x) \xrightarrow[n\to\infty]{} \cos^2(\tfrac{1}{2}\theta) m_\theta$$

almost everywhere on $c(y, \theta)$, where χ_{A_j} is the indicator function of A_j. Thus the relative frequencies conform with the density formula.

It is natural at this stage to look for a probability measure on $\mathfrak{F}_0$ such that the observed frequencies would correspond with the measure of the set $\{s \in \mathfrak{F}_0 \mid s(x) = \frac{1}{2} \text{ and } s(y) = \frac{1}{2}\}$. As noted above no such measure exists and we are therefore left with a nonclassical concept of probability.

Up to this point the model achieves no more than previous proposals in the literature[8] for recovering the quantum statistics of successive measurements on a *single* spin-$\frac{1}{2}$ particle by defining a "spin function" which assumes a precise value in any direction. These models, in effect, simulate interference effects by assuming that a measurement disturbs the spin function in a definite (though unknown) way. For precisely this reason, they are unable to account for the statistical correlations of coupled systems without introducing a nonlocal mechanism for transforming a measurement disturbance instantaneously from one system to the other. In the model I have proposed a measurement of the spin in one direction does change the spin value in various different directions (by rotating or reflecting the spin function) but it does not introduce a *statistical* disturbance. In my view *the essence of the Einstein-Podolsky-Rosen "paradox" is that it shows that the conditional probabilities are, in fact, invariant.*

From the Pauli exclusion principle, we know that two interacting electrons emerge from the interaction in the singlet state. In our terminology, this means after the interaction the electrons have opposite spins in all directions, that is $s_1(x) = -s_2(x)$ where s_1 and s_2 are the corresponding spin functions. Hence one can gain knowledge of the values of s_1 in two different directions x, y by directly measuring $s_1(x)$ and inferring the valus of $s_1(y)$ from a measurement of $s_2(-y)$. If the electrons are sufficiently separated when the measurements are performed, we can assume that they do not interfere with one another. (This is the principle of locality.) Suppose that we have a source of electron pairs that emits the electrons of each pair with opposite spin functions and in opposite directions. We get two beams that travel in opposite directions, the electron spin functions in each beam are randomly oriented, and the spin values of a given electron in the right-hand beam are oppositely correlated with the spin values of its pair electron in the left-hand beam. The relative frequency of the event "spin up in the x direction for electron 1 and spin down in the y direction for electron 2" is $\frac{1}{2}\cos^2(\frac{1}{2}\theta)$ and thus the conditional probability of "spin up in the x direction for electron 1 given that the spin is up in the y direction for *electron* 1" is $\cos^2(\frac{1}{2}\theta)$. Since in the present model, the measurements establish the simultaneous spin values of a single electron in two directions, they serve as a verification of the density formula (1) and also show it to be invariant.

The interpretation of joint probabilities as density conditions on circles, and the fact that these conditions are invariant, is a unique feature of the proposed model. The invariance is a key feature since without it one cannot explain (locally) the fact that the relative frequencies of successive measurements on a single electron are numerically identical to electron pair correlations. This is the reason why this model is the first to account for the Einstein-Podolsky-Rosen experiment without violating the principle of locality. Indeed, all the above observations are based on the validity of the principle of locality, that is, the statistical independence of measurements performed on electron pairs in the singlet state. The relative frequencies violate Bell's inequality the way they do *because* the locality principle is *true*. The violation of Bell's inequality reflects a *mathematical truth*, namely, that certain density conditions are incompatible with the existing theory of probability.

Conceptually this is the crucial point. The standard explanations of Bell's paradox all assume the validity of probability theory together with its definitions of statistical independence and conditionalization. It is believed, in other words, that Kolmogorov's axioms ultimately capture what we mean by the term "probability." This attitude resembles Kant's view of Euclidean geometry as an *a priori* synthetic truth. There are good independent mathematical reasons for extending the concept of probability and there seem to be good physical reasons as well. I suggest treating the theory of probability in the same manner that Riemann and Einstein treated geometry.

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[1]J. S. Bell, Physics (Long Island City, N.Y.) 1, 195–199 (1964).

[2]E. P. Wigner, Am. J. Phys. 38, 1005 (1970).

[3]J. F. Clauser and M. A. Horne, Phys. Rev. 10, 526 (1974).

[4]For a survey of experimental results, see J. F. Clauser and A. Shimony, Rep. Prog. Phys. 49, 349–358 (1978).

[5]M. E. Rudin, in *Handbook of Mathematical Logic*, edited by J. Barwise (North-Holland, Amsterdam, 1977), see in particular, theorem 15 on p. 498.

[6]Let $f:[0,\pi]\to[0,1]$ be any function such that $f(0)=1$ and $f(\theta)+f(\pi-\theta)=1$. One can generalize the existence theorem and show that there are spin functions that satisfy a density formula which is like (1) except that we replace $\cos^2(\frac{1}{2}\theta)$ and $\sin^2(\frac{1}{2}\theta)$ by $f(\theta)$ and $1-f(\theta)$, respectively. By an iteration of Bell's inequality one can prove that the density values given by $f(\theta)$ are representable as conditions on a probability space only if $f(\theta_j)\leq 1-\theta_j/\pi$ for $\theta_j=\pi/2^{j+1}$, $j=1,2\ldots$. The function $1-\theta/\pi$ is sometimes referred to as "the Bell limit."

[7]On Wigner rotation, see W. M. Gibson and B. R. Pollard, *Symmetry Principles in Elementary Particle Physics* (Cambridge Univ. Press, Cambridge, England, 1976), pp. 111–117.

[8]See Bell (Ref. 1); J. F. Clauser, Am. J. Phys. 39, 1095 (1971); S. Kochen and E. P. Specker, J. Math. Mech. 17, 59 (1967).