

## Magnetic Flux, Angular Momentum, and Statistics

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It is demonstrated that the orbital angular momentum  $l_z$  of a particle of charge  $q$  orbiting around a tube with magnetic flux  $\Phi$  is quantized in units  $l_z = \text{integer} - q\Phi/2\pi$ . A very simple physical argument for this is presented, and applied to understand the Dirac quantization condition and the charge-spin relation for particles bound to magnetic monopoles. The unusual statistics of flux-tube-charged-particle composites is discussed.

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The unusual angular momentum of magnetic-monopole-charged-particle composites has been recognized for a long time<sup>1</sup> and has recently been demonstrated also in the framework of unified gauge theories.<sup>2</sup> The main result is that composites satisfying the Dirac quantization condition  $gq/4\pi = \frac{1}{2} + \text{integer}$  ( $g$  is the magnetic charge and  $q$  is the electric charge) have angular momentum shifted by  $\frac{1}{2}$  from the naive value—e.g., composites of a spinless monopole and a scalar particle satisfying the Dirac equation will have  $J = \text{integer} + \frac{1}{2}$ . It has also been shown, surprisingly recently, that the statistics of these composites is effectively modified so as to restore the usual spin-statistics connection.<sup>3,4</sup>

Although in three spatial dimensions the angular momentum can only be integer or half integer, in an effectively two-dimensional situation (e.g., in the presence of a string configuration) more possibilities open up, both for the spin and for the statistics. Also, the lower dimensional system should be simpler to understand. It therefore may be interesting to reexamine these questions in the two-dimensional context. This note presents the results of such a reexamination. As we shall see interesting phenomena do occur in the two-dimensional case; furthermore it becomes possible, following a suggestion of Johnson, to understand all the phenomena (in both two and three dimensions) in an extremely simple physical way.<sup>5</sup>

(A) Imagine a particle of charge  $q$  which orbits around, but does not penetrate, a solenoid running along the  $z$  axis. When no current flows through the solenoid the orbital angular momentum is of course quantized in units  $l_z = \text{integer}$ . If a current is slowly turned on, the charged particle will feel an electric field  $\vec{E}(\vec{r}) = -[2\pi(x^2 + y^2)]^{-1}\hat{z} \times \vec{r}\dot{\Phi}$  according to Faraday's law, where  $\dot{\Phi}$  is the flux through the solenoid. This gives the change in angular momentum

$$\dot{l}_z = [\vec{r} \times (q\vec{E})]_z = -(q/2\pi)\dot{\Phi}. \quad (1)$$

The total change in angular momentum therefore depends only on the final flux,  $\Delta l_z = -q\Phi/2\pi$ . The quantized angular momenta are correspondingly displaced, and so we conclude that when a flux  $\Phi$  runs through the solenoid, the angular momentum  $l_z$  is quantized in units

$$l_z = \text{integer} - q\Phi/2\pi.$$

(B) The same conclusion may be reached more formally as follows. Although the magnetic field vanishes outside the solenoid of course the potential does not. Rotations around the  $z$  axis, i.e., changes in the azimuthal angle  $\varphi$ , are generated by the covariant angular momentum  $l_z = -i\partial_\varphi - qA_\varphi$ . In a nonsingular gauge,  $A_\varphi = \Phi/2\pi$  and the azimuthal dependence of the electron wave function is  $\psi_n \propto e^{in\varphi}$ ,  $n = \text{integer}$  for continuity. Then  $l_z\psi_n = (n - q\Phi/2\pi)\psi_n$ , in agreement with the previous conclusion.

(C) Finally one may eliminate the potential outside the solenoid altogether by a singular gauge transformation from (B), namely,  $\vec{A}' = \vec{A} - \nabla\Lambda$  with  $\Lambda = \Phi\varphi/2\pi$ . This is singular because  $\varphi$  is a multivalued function. The charged-particle wave function  $\psi$  now obeys a free Schrödinger equation but with an unusual boundary condition that  $\psi'(\varphi + 2\pi) = e^{-iq\Phi}\psi'(\varphi)$  following from the gauge transformation  $\Lambda$ . This boundary condition requires  $\psi'(\varphi) \propto \exp[i(\text{integer} - q\Phi/2\pi)\varphi]$ . Now there is no vector potential, and the angular momentum is identified as usual, so that  $l_z = \text{integer} - q\Phi/2\pi$ .

If we interchange flux-tube-charged-particle composites we will have, in addition to the usual factors, a phase factor appearing in all gauge-invariant observables.<sup>3</sup> This is because, in the motion depicted in Fig. 1, each composite must be covariantly transported in the gauge potential of the other. The resulting phase is simply  $e^{iq\Phi}$  (since  $A_\varphi = \Phi/2\pi$ ).

If  $l_z = \text{integer} - q\Phi/2\pi$  is an integer, the phase factor is unity and the statistics is normal (e.g., if the composite is a flux tube plus electron it

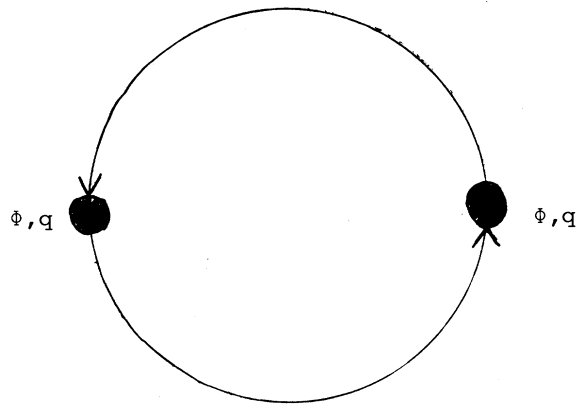


FIG. 1. Interchange of two flux-tube-particle composites by a rotation.

obeys Fermi statistics). If  $l_z = \text{integer} - q\Phi/2\pi$  is half an odd integer then the normal statistics is reversed. In intermediate cases, the composites cannot be described as fermions nor as bosons. In a scattering experiment the direct and exchange terms will interfere with a coefficient  $\cos(q\Phi)$ . Notice that in this case the possible ambiguity of the sign of the phase is unimportant (i.e., we can reverse the direction in Fig. 1). This is not true for assemblages of more than two of these particles, which are therefore complicated to describe.

Some applications of the above discussions are now presented.

(a) *Superconducting vortex*.—For a unit vortex in a superconductor the magnetic flux is quantized as  $\Phi = 2\pi/2e$ , where  $e$  is the electron charge (and  $2e$  the charge of the condensate). Therefore the orbital angular momentum of an electron around a unit vortex is  $\frac{1}{2} + \text{integer}$  and the composite is a boson.

(b) *Dirac condition*.—We can repeat the Faraday law argument (A) above as we imagine turning on a magnetic monopole (Fig. 2). For a charged particle orbiting infinitesimally above the pole the flux is  $g/2$  while for a particle infinitesimally below it is  $-g/2$ . The corresponding shifts in angular momentum are  $l_z \mp gq/4\pi$ ; the allowed  $l_z$  values are  $\text{integer} \mp gq/4\pi$ . For these two spectra of  $l_z$  values to agree one must have  $gq/4\pi = \frac{1}{2} + \text{integer}$ , which is the Dirac quantization condition.

(c) *Charge-spin relation*.—As a consequence of this argument when  $gq/4\pi$  is half odd integer the angular momentum spectrum has been shifted by  $\frac{1}{2}$  unit. I find this explanation of the relationship between the Dirac condition and the spin shift of

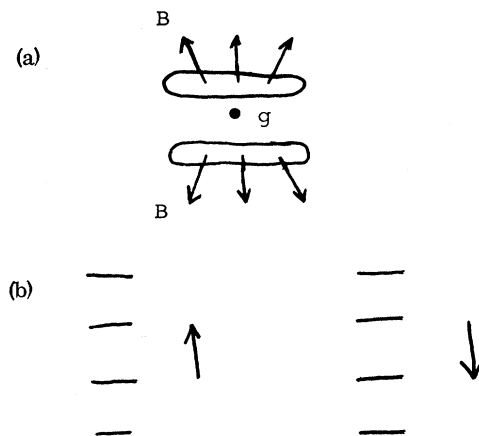


FIG. 2. (a) Sign of magnetic flux depends on whether orbit is infinitesimally above or below the pole. (b) Spectrum of angular momenta moves up or down as the flux is turned on. They can meet after each has shifted by  $\frac{1}{2}$  unit, explaining the Dirac quantization condition and the unusual spin of composites.

$\frac{1}{2}$  unit, in terms of the  $l_z$  spectrum meeting itself after up and down shifts, very appealing.

(d) *Gauge theory strings*.—Consider a U(1) gauge theory coupled to a complex Higgs field  $h$  of charge  $nq$  and another matter field with the fundamental charge  $q$ . There are topologically stable vortices or strings where the order parameter has the asymptotic behavior

$$\langle h(r, \varphi) \rangle \xrightarrow{r \rightarrow \infty} v e^{i\varphi}.$$

As  $\varphi$  varies the local values of the order parameter are related by a gauge rotation through  $\varphi/n$ , since for the charge- $n$  field  $h$  we have  $h \rightarrow e^{in(\varphi/n)}h$  under such a rotation. Arguments as above show that particles of charge  $q$  orbiting around such a string have orbital angular momentum  $l_z = 1/n + \text{integer}$ .

This sort of setup can occur in non-Abelian models used in attempts to unify strong, electromagnetic, and weak interactions. As a toy example consider an SU(2) gauge theory broken completely by a three-index symmetric tensor  $\langle h_{ijk} \rangle_0 = v \delta_{ij} \delta_{jk} \delta_{ki}$ . Actually, it is important for us that more precisely SU(2) is broken to Z(3), the Z(3) group being generated by the transformation  $\exp(\frac{2\pi}{3}iI_3)$ . We can have a stable string because if the order parameter has the asymptotic behavior

$$\langle h_{ijk}(r, \varphi) \rangle \xrightarrow{r \rightarrow \infty} v e^{i\varphi} \delta_{ij} \delta_{jk} \delta_{ki},$$

then covariant transport from  $\varphi = 0$  to  $\varphi = 2\pi$  gen-

erates a nontrivial  $Z(3)$  transformation which cannot be continuously deformed away. Bound states of an  $SU(2)$  vector matter field in the presence of this string will have  $l_z = \frac{1}{3} + \text{integer}$ .

(e) *Thought experiment.*—The essentially classical nature of all these effects (given quantized angular momentum for a *free* particle), and the measurability of the fractional angular momentum, is brought out by the following simple thought experiment. Consider a long but finite solenoid along the  $z$  axis and a charged electron coming up at it from below. As the particle moves up it will feel a torque from the fringing fields, which of course will generate the angular momentum  $-q\Phi/2\pi$  as before. The solenoid will rotate to compensate, and the rate of rotation could be used to measure  $l_z$ .

(f) *Dyon charge.*—There is a striking formal analogy between the fractional angular momentum  $l_z = \text{integer} - q\Phi/2\pi$  and the electric charge in dyons in a  $\theta$  vacuum,<sup>6</sup>  $q/e = \text{integer} - \bar{g}\theta/2\pi$ , where  $\bar{g}$  is the strength of the magnetic pole in units of the Dirac pole  $2\pi/e$ . Mathematically, this comes about as follows. The above arguments, in particular (B), can be summarized in the statement that in a background of flux  $\Phi$  (in the presence of a magnetic charge  $g$ ) the rotational symmetry requires that rotation in physical space through angle  $\varphi$  must be accompanied by a gauge rotation through  $-\varphi\Phi/2\pi$  ( $-\varphi\bar{g}/2\pi$ ). The  $\theta$  vacuum is de-

fined by the analogous requirement that rotations in gauge space through  $\varphi$  should be accompanied by multiplication with the phase  $-\varphi\bar{g}\theta/2\pi$ , which can be thought of as a rotation in *magnetic* gauge space by  $-\varphi\theta/2\pi$ . This intimate relation among rotations in physical, ordinary gauge, and magnetic gauge spaces gives the way to understanding dyon statistics and is further explored in the accompanying note.<sup>4</sup>

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<sup>1</sup>M. N. Saha, *Indian J. Phys.* **10**, 145 (1936), and *Phys. Rev.* **75**, 1968 (1949); M. Fierz, *Helv. Phys. Acta* **17**, 27 (1944); H. A. Wilson, *Phys. Rev.* **75**, 309 (1949).

<sup>2</sup>R. Jackiw and C. Rebbi, *Phys. Rev. Lett.* **36**, 1116 (1976); P. Hassenfratz and G. 't Hooft, *Phys. Rev. Lett.* **36**, 1119 (1976).

<sup>3</sup>A. Goldhaber, *Phys. Rev. Lett.* **36**, 1122 (1976).

<sup>4</sup>F. Wilczek, following Letter [*Phys. Rev. Lett.* **48**, 1146 (1982)] (for dyons of fractional charge).

<sup>5</sup>Peculiar features of the Schrödinger equation for an electron in the presence of a vortex in a superconductor have been noticed for some time. See A. Fetter and P. P. Hohenberg, in *Superconductivity*, edited by R. D. Parks (Dekker, New York, 1968), p. 688.

<sup>6</sup>E. Witten, *Phys. Lett.* **86B**, 283 (1979).

## Remarks on Dyons

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Some issues in dyon theory are discussed: retention of the charge  $e\theta/2\pi$  in the chiral limit, the spin-statistics connection, and the indefiniteness of anomalous charges.

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In a profound paper Witten<sup>1</sup> demonstrated that in a  $\theta$  vacuum magnetic poles will necessarily be dyons, with charge  $e\theta/2\pi + \text{integer}$  for the unit pole. This result raises several apparent puzzles, and an examination of its demonstration indicates that in theories with fermions the dyons will have a rich internal structure.

It is widely believed that all effects of the  $\theta$  parameter disappear when there is an appropriate anomalous chiral symmetry in the problem. In-

deed the anomaly equation  $\partial_\mu j_\mu^5 \propto \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$  indicates, according to Noether's theorem, that we can add or subtract terms proportional to  $\epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$  from the Lagrangian of the theory by making a chiral rotation. Since  $e^2\theta/16\pi^2$  is defined as the coefficient of  $\epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$  in the Lagrangian, this would seem to indicate that all the theories with different values of  $\theta$  are physically equivalent, related simply by redefinition of the fermion fields. This argument poses a puzzle: