Fluctuation Properties of Nuclear Energy Levels: Do Theory and Experiment Agree?

R. U. Haq
Physique Théorique et Mathématique, Université Libre de Bruxelles, B-1050 Brussels, Belgium, and Department of Physics, University of Toronto, Toronto M5S 1A7, Canada

and

A. Pandey
Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay, Cédex, France, and Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

and

O. Bohigas
Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay, Cédex, France
(Received 11 December 1981)

The fluctuation properties of nuclear energy levels are analyzed with new spectrally averaged measures. A remarkably close agreement between the predictions of random-matrix theories and experiment is found.

PACS numbers: 21.10.Ma, 24.60.+m

Since the pioneering work of Wigner and its systematic development by many authors, it has become clear that random-matrix theories[1,2] provide a natural theoretical framework for nuclear energy-level fluctuations, i.e., departures from spectral uniformity. The Gaussian orthogonal ensemble (GOE) of asymptotically large real symmetric random matrices is the model that has been most extensively investigated. Almost twenty years ago Dyson and Mehta[3] emphasized one of the main questions at issue: "We would be very happy if we could report that our theoretical model had been strikingly confirmed by the statistical analysis of neutron capture levels. We would be even happier if we could report that our theoretical model had been decisively contradicted.... Unfortunately, our model is as yet neither proved nor disproved." A large amount of effort has been devoted since then to providing accurate data on slow-neutron resonances of heavy nuclei as well as on proton resonances of light nuclei, and to their analysis. Despite this, only relatively vague, although significant, "overall good agreement" between theory and experiment can be quoted from subsequent work on the subject. It is the purpose of this Letter to settle once and for all this question for energy-level fluctuations.

Our analysis will differ from the previous ones in two main respects: (i) The *whole body of high-quality data* is analyzed simultaneously, and (ii) several *new spectral measures* are introduced. In particular, special care regarding sample-size effects can then be taken when comparing data with predictions. Use will be made of what is known analytically, supplemented with new Monte Carlo results when analytic expressions are not available.

In the conventional analysis one treats the data from each nucleus, containing about 100 levels, separately and consequently the statistical significance is poor. On the other hand, one expects that the resonance energies of every compound nucleus will share the same fluctuation properties and it seems therefore natural, and in the spirit of random-matrix theory, to treat the combined set of nuclear resonance-energy data of different nuclei—in short, the nuclear data ensemble (NDE)—as a sampling of eigenvalues of GOE matrices; see Rosenzweig and Porter[4] for a similar notion for complex atomic spectra. The data analyzed here consist of 1407 resonance energies corresponding to 30 sequences of 27 different nuclei: (i) slow-neutron resonance data[5] on ^{110,112,114}Cd, ^{152,154}Sm, ^{154,156,158,160}Gd, ^{160,162,164}Dy, ^{166,168,170}Er, ^{172,174,176}Yb, ^{182,184,186}W, ^{232}Th, and ^{238}U (1146 levels); (ii) proton resonance data[6] on ^{44}Ca ($J=\frac{1}{2}^-$), ^{44}Ca ($J=\frac{1}{2}^+$), and ^{48}Ti ($J=\frac{1}{2}^+$) (157 levels); and (iii) (n,γ) reaction data[7] on ^{177}Hf ($J=3$), ^{177}Hf ($J=4$), ^{179}Hf ($J=4$), and ^{179}Hf ($J=5$) (104 levels). The relative scarcity of data useful for our purpose is due to the very stringent conditions that are imposed. For each nucleus the sequence should ideally be *complete* (no missing levels) and *pure* (the same angular momentum and parity).

Fluctuation properties are in general characterized by the set of ν-level cluster functions[8] Y_ν ($\nu \geq 2$). Although all cluster functions have been evaluated[9] for the GOE, mostly measures deriving from Y_2 only have been used so far. There are several reasons for this: One should follow a hierarchy and start with $\nu = 2$; the higher the value of ν, the more complicated the structure of Y_ν, and therefore the harder it is to produce meaningful measures that can be derived analytically; moreover, detection of higher-order effects requires larger samples. For properties related to Y_ν with $\nu > 2$, we shall mostly rely on Monte Carlo results that we have obtained from a sample, adequate for our purpose, of 100 GOE matrices of dimensionality 300.

To begin with, we consider the function Y_2. For the GOE,[8]

$$Y_2(r) = \left(\frac{\sin\pi r}{\pi r}\right)^2 + \left(\int_r^\infty \frac{\sin\pi r'}{\pi r'}dr'\right)\frac{d}{dr}\left(\frac{\sin\pi r}{\pi r}\right), \quad (1)$$

which has the value unity at $r=0$ and goes to zero as $(\pi r)^{-2}$ for large r ($\gtrsim 1$). However, since the large-r form cannot be tested accurately from the data, one considers measures which involve integrations over Y_2. One such measure[2,3] is $\Sigma^2(\bar{n})$, the variance of the number n of levels in a given interval of length[10] $\bar{n}$:

$$\Sigma^2(\bar{n}) = \bar{n} - 2\int_0^{\bar{n}} (\bar{n} - r)Y_2(r)dr$$
$$= (2/\pi^2)\{\ln(2\pi\bar{n}) + \gamma + 1 + \tfrac{1}{2}[\mathrm{Si}(\pi\bar{n})]^2 - \tfrac{1}{2}\pi\,\mathrm{Si}(\pi\bar{n}) - \cos(2\pi\bar{n}) - \mathrm{Ci}(2\pi\bar{n}) + \pi^2\bar{n}[1 - (2/\pi)\mathrm{Si}(2\pi\bar{n})]\}. \quad (2)$$

Another is the Δ_3 statistic of Dyson and Mehta[3] (higher moments of Δ_3 involve functions with $\nu > 2$ as well). It measures, for a fixed interval $[x, x+\bar{n}]$, the least-squares deviation of the staircase $N(E)$, the number of levels with energy less than or equal to E, from the best straight line fitting it:

$$\Delta_3(\bar{n};x) = (1/\bar{n})\min_{A,B}\int_x^{x+\bar{n}} [N(E) - AE - B]^2 dE. \quad (3)$$

Its ensemble average is related[11] to Σ^2, and hence to Y_2, by

$$\bar{\Delta}_3(\bar{n}) = (2/\bar{n}^4)\int_0^{\bar{n}} (\bar{n}^3 - 2\bar{n}^2 r + r^3)\Sigma^2(r)dr, \quad (4)$$

which may be integrated numerically; see Fig. 1(a). Note that the ensemble properties, due to stationarity,[11] are independent of the position x of the interval $[x, x+\bar{n}]$ and that the asymptotic-$\bar{n}$ results[3] [e.g., $\bar{\Delta}_3(\bar{n}) \simeq \pi^{-2}\ln\bar{n} - 0.007$, $\mathrm{Var}\Delta_3 \simeq 0.012$], the only ones used so far to analyze data, may not be of much interest for small $\bar{n}$; see Fig. 1(b).

We have made an extensive test of the theory by studying the aforementioned functions $Y_2(r)$, $\Sigma^2(r)$, and $\Delta_3(\bar{n})$ as well as others (spacing distributions, etc.). A complete account will be given elsewhere and from now on we shall restrict ourselves to Δ_3. This choice is sensible because Δ_3 is adapted to describe one of the most characteristic features of the GOE (the spectral rigidity) and also because it provides among the sharpest comparisons between theory and experiment.

In order to make optimal use of the data, we introduce *spectral-averaged quantities*. Consider a sequence containing p levels. As an example consider the spectral average

$$\langle\Delta_3(\bar{n})\rangle_p = (p - \bar{n})^{-1}\int_x^{x+(p-\bar{n})} \Delta_3(\bar{n}, y)dy \quad (5)$$

which estimates $\bar{\Delta}_3(\bar{n})$. Such an estimate will be meaningful only if sample errors due to finite p are also given. The variance of $\langle\Delta_3\rangle_p$, which is the square of the sample error, is given by

$$\mathrm{Var}\langle\Delta_3(\bar{n})\rangle_p \equiv \overline{\langle\Delta_3(\bar{n})\rangle_p{}^2} - \overline{\langle\Delta_3(\bar{n})\rangle_p}^2 = \frac{2\mathrm{Var}\Delta_3(\bar{n})}{(p-\bar{n})^2}\int_0^{p-\bar{n}} (p - \bar{n} - r)C(r;\bar{n})dr \simeq \frac{2\mathrm{Var}\Delta_3(\bar{n})}{p}\int_0^1 C(\lambda\bar{n};\bar{n})d\lambda. \quad (6)$$

In (6) $C(r;\bar{n})$ is the autocorrelation function of Δ_3:

$$C(r;\bar{n}) = [\mathrm{Var}\Delta_3(\bar{n})]^{-1}\{\overline{\Delta_3(\bar{n};x)\Delta_3(\bar{n};x+r)} - [\bar{\Delta}_3(\bar{n};x)]^2\}. \quad (7)$$

The last step of Eq. (6), which has been checked by Monte Carlo calculations, is valid for $p \gg \bar{n} \gtrsim 2$. We emphasize that it is the theoretical model itself, with which one compares data, that provides the sample errors. Notice moreover that as $p \to \infty$, $\mathrm{Var}\langle\Delta_3(\bar{n})\rangle_p$ goes to zero (ergodic property[11]). In practice, instead of (5) we will use a less smooth average in which the integral is replaced by a summation.

We now turn to the data (NDE). We calculate for each of the thirty sequences the spectral-averaged quantity $\langle\Delta_3(\bar{n})\rangle_p$ and then take their average, weighted according to the size of each se-

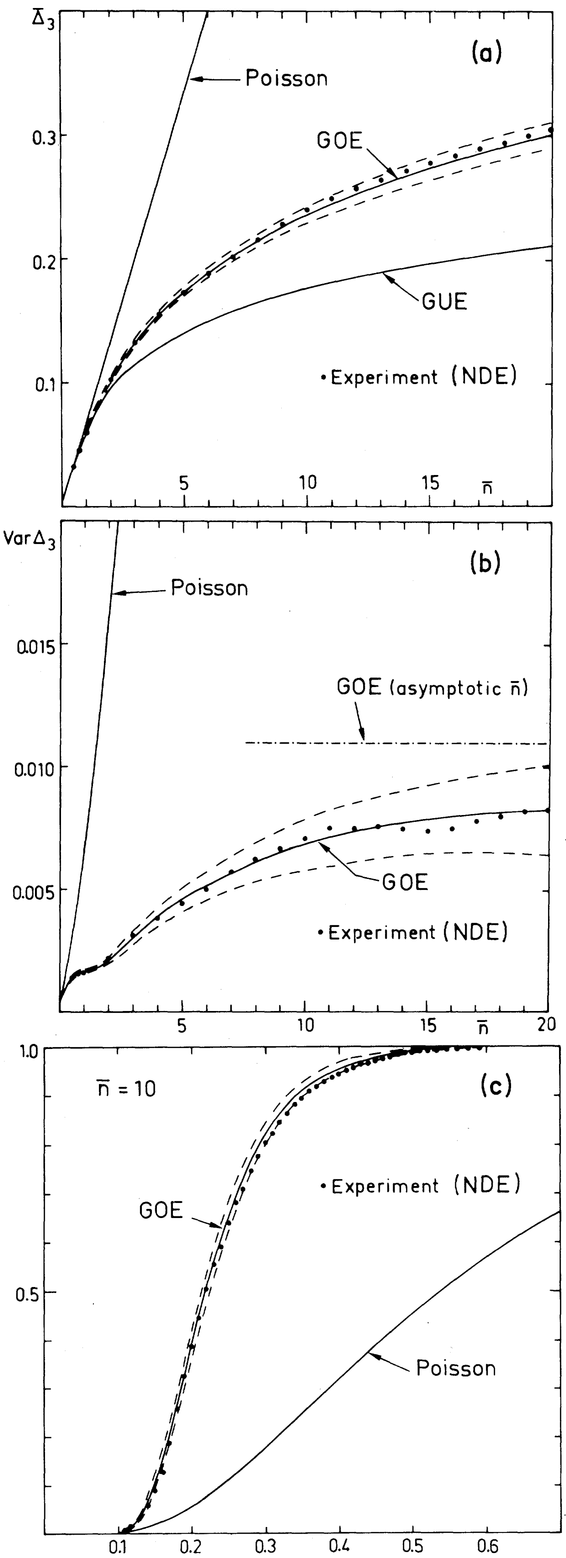

FIG. 1. (a) $\bar{\Delta}_3$ and (b) VarΔ_3 as functions of $\bar{n}$; (c) distribution function of Δ_3 for $\bar{n} = 10$. Dashed lines correspond, for GOE, to one standard deviation from the average.

quence. We have considered the range $0 \leqslant \bar{n} \leqslant 100$. The results for $\bar{n} \leqslant 20$ are shown in Fig. 1(a). They are in a remarkably good agreement with the GOE. We emphasize that typically the "figure of merit" $(\text{Var}w)^{1/2}/\bar{w}$ for a measure w is of the order of a few percent in the present analysis, whereas in the conventional one it is ~20% or more. That the data points for various values of $\bar{n}$ all lie within one sample error is partly due to the fact that they are not totally independent. Furthermore, since $\langle\Delta_3(\bar{n})\rangle$ is a smooth function of $\bar{n}$, there is no need to insert more points in the range considered.

To explore effects of $\nu > 2$ we have also considered the ensemble variance of $\Delta_3(\bar{n})$ and the distribution of Δ_3 for various values of $\bar{n}$. We proceed in the same way as before except that now the GOE predictions come from Monte Carlo calculations. The results for VarΔ_3 are shown in Fig. 1(b) and one example ($\bar{n} = 10$) of the distribution function of Δ_3 is shown in Fig. 1(c). The agreement between GOE predictions and data (NDE) is again very good. The values corresponding to the Poisson case (a spectrum which unlike the GOE has no level repulsion nor any correlation between spacings) are also given for comparison.

We have thus established an astonishingly good agreement between a parameter-free theory (GOE) and the data. We emphasize that, apart from *rotation and time-reversal invariance* resulting in the real symmetric nature of the matrices, the GOE takes no account of the specific properties of the nuclear Hamiltonian, e.g., its (1 + 2)-body nature, its large pairing and quadrupole components, etc. We also recall that eigenvalues of realistic (nonrandom) nuclear shell-model matrices show the same fluctuation patterns.[2,12] How can this be understood theoretically? Monte Carlo calculations[12] have indicated that ensembles of two-body operators acting in many-particle (shell-model) spaces also yield fluctuations characteristic of the GOE. More recently[13] it has been shown that adding a GOE matrix H to any real symmetric matrix K $(K + \alpha H)$ leads very quickly, as α increases, to the same fluctuations. Intermediate fluctuation patterns are to be expected only when the random-matrix elements are not much larger than the local average spacing of the given (nonrandom) matrix K. The good agreement with experiment, coupled with the theoretical understanding which is slowly emerging, reinforces the belief that the GOE fluctuations are to be found in nature under very general conditions.

We mention finally that one can use the close agreement to impose restrictions on mechanisms which would change the fluctuations. In particular if time reversal is not an exact symmetry, the appropriate model would be an ensemble of complex Hermitian matrices, in which the real and imaginary parts are sampled independently. {The Gaussian unitary ensemble [GUE, see Fig. 1(a)] in which the two parts have the same norm, is one such example.} But even a small magnitude of the imaginary part induces major changes in the fluctuation properties. This notion is being pursued[14] to derive an upper bound on the time-reversal–noninvariant part of the nuclear Hamiltonian.

We are grateful to J. B. French for valuable suggestions and a critical reading of the manuscript. We also acknowledge helpful discussions with S. S. M. Wong. One of us (A.P.) was a Fellow of the Fondation Joliot-Curie, 1980–1981. The Division de Physique Théorique del' Institut de Physique Nucléaire is a Laboratoire associé au Centre National de la Recherche Scientifique.

[1] *Statistical Theories of Spectra: Fluctuations*, edited by C. E. Porter (Academic, New York, 1965); M. L. Mehta, *Random Matrices and the Statistical Theory of Energy Levels* (Academic, New York, 1967).

[2] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, Rev. Mod. Phys. 53, 385 (1981); this review covers recent developments and contains up-to-date references in the field.

[3] F. J. Dyson and M. L. Mehta, J. Math. Phys. 4, 701 (1963).

[4] N. Rosenzweig and C. E. Porter, Phys. Rev. 120, 1698 (1960).

[5] H. I. Liou, G. Hacken, J. Rainwater, and U. N. Singh, Phys. Rev. C 11, 462 (1975), and earlier references therein; for ^{156}Gd, see C. Coceva and M. Stefanon, Nucl. Phys. A315, 1 (1979), and private communication.

[6] W. M. Wilson, E. G. Bilpuch, and G. E. Mitchell, Nucl. Phys. A245, 285 (1975).

[7] C. Coceva, F. Corvi, P. Giacobbe, and M. Stefanon, in *Statistical Properties of Nuclei*, edited by J. B. Garg (Plenum, New York, 1972).

[8] F. J. Dyson, J. Math. Phys. 3, 166 (1962).

[9] F. J. Dyson, Commun. Math. Phys. 19, 235 (1970); M. L. Mehta, Commun. Math. Phys. 20, 245 (1971).

[10] For a random variable w, we denote its average by $\overline{w}$ and its ensemble variance $\overline{(w-\overline{w})^2}$ by Var$w$. Note, moreover, that we are considering spectra normalized to unit local average spacing.

[11] A. Pandey, Ann. Phys. (N.Y.) 119, 170 (1979).

[12] S. S. M. Wong and J. B. French, Nucl. Phys. A198, 188 (1972); O. Bohigas and M. J. Giannoni, Ann. Phys. (N.Y.) 89, 393 (1975).

[13] A. Pandey, Ann. Phys. (N.Y.) 134, 110 (1981).

[14] J. B. French, V. K. B. Kota, and A. Pandey, to be published.

Direct Observation of Nonequilibrium Effects in Sequential Fission

P. Glässel, D. v. Harrach, L. Grodzins,[a] and H. J. Specht
Physikalisches Institut der Universität Heidelberg, D-6900 Heidelberg, Germany
(Received 18 February 1982)

Kinematically complete experiments have been performed on the three-body exit channels in 12.5-MeV/u ^{129}Xe on ^{122}Sn. The events observed result from a sequential fission process taking place in 1×10^{-21} s. The angular distribution of the fission axis is approximately collinear with the axis of the first scission, and the mass distribution of the fission is asymmetric with the heavier mass preferentially emitted opposite to the direction of the third particle. These effects show the existence of a new phenomenon of nonequilibrium fission.

PACS numbers: 25.70.Bc, 25.85.Ge

The three-body exit channels observed in nuclear reactions between heavy ions at energies below 12.5 MeV/u have thus far successfully been interpreted as two-step processes.[1,2] In the first binary step, two highly excited rotating fragments are formed by the transfer of kinetic energy and angular momentum from the relative motion; in the second step, one of the two resulting compound nuclei undergoes subsequent fission. Recently, however, unusually high fission probabilities, not compatible with equilibrium fission from a compound nucleus, have been reported for intermediate systems with masses of 100-150 u, formed in 12.1-MeV/u ^{84}Kr on ^{166}Er,[3] and 12.5-MeV/u ^{84}Kr on ^{90}Zr, ^{166}Er and ^{129}Xe on ^{122}Sn.[4] In Ref. 4, moreover,

