Reduction of Dynamical Degrees of Freedom in the Large-N Gauge Theory

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It is pointed out that the factorization of disconnected Wilson loop amplitudes implies a major reduction in the dynamical degrees of freedom in the large-N limit of lattice gauge theory; the original model may be replaced by a much simpler one (d is the space-time dimensionality),

$$Z = \prod_\mu \int dU_\mu \exp(\beta \sum_{\mu \neq \nu=1}^{d} \operatorname{tr} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger).$$

Thus the field theory may be reduced to an integration over a finite number of matrices in large-N limit.

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Currently the lattice formulation of gauge theory[1] appears to provide the most systematic approach to testing the idea of quark confinement. Unfortunately lattice gauge models seem to possess an extraordinary complexity and have so far defied attempts at their analytic solutions. Here one may hope that in the limit of large N a considerable simplification might take place in U(N) [or SU(N)] gauge theory because of the dominance of planar surfaces. It has been pointed out some time ago[2] that the planar approximation implies the factorization of Wilson loop amplitudes for disconnected loops and this may give us a clue to the understanding of the properties of large-N gauge models.

In this paper we shall show that in fact the large-N factorization implies a remarkable simplification in the structure of the theory; the standard U(N) gauge theory defined by the partition function

$$Z = \prod_x \prod_\mu \int dU_{x,\mu} \exp\{\beta \sum_x \sum_{\mu \neq \nu=1}^{d} \operatorname{tr} U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu}^\dagger U_{x,\nu}^\dagger\} \tag{1}$$

can be effectively replaced by a much simpler model,

$$Z_r = \prod_\mu \int dU_\mu \exp\{\beta \sum_{\mu \neq \nu=1}^{d} \operatorname{tr} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\} \tag{2}$$

in the limit $N \to \infty$ with $\lambda = N/\beta$ kept fixed. Here $U_{x,\mu}$ is an $N \times N$ unitary matrix lying on a link (x, μ) which connects lattice sites x and $x+\mu$. μ denotes a unit vector in the μ direction and d is the space-time dimensionality. Our model is obtained from the standard one by identifying all variables on the links in the same direction, $U_{x,\mu} = U_\mu$. Only global invariance is left in the theory and we shall call it the reduced model.

Corresponding to a Wilson loop amplitude in the standard model,

$$W(C) = \langle \operatorname{tr}(U_{x,\mu} U_{x+\mu,\nu} U_{x+\mu+\nu,\alpha} \cdots U_{x-\sigma,\sigma}) \rangle, \tag{3}$$

one considers an amplitude

$$W_r(C) = \langle \operatorname{tr}(U_\mu U_\nu U_\alpha \cdots U_\sigma) \rangle, \tag{4}$$

where expectation values in Eqs. (3) and (4) are taken with respect to the weights in Eqs. (1) and (2), respectively. Here the point is that to a given sequence of indices $(\mu, \nu, \alpha, \ldots, \sigma)$ one may associate a unique (modulo overall translation) contour connecting lattice sites $(x, x+\mu, x+\mu+\nu, x+\mu+\nu+\alpha, \ldots, x+\mu+\nu+\ldots+\sigma)$ in succession and vice versa and we have the identification

$$C: (\mu, \nu, \alpha, \ldots, \sigma) \sim (x, x+\mu, x+\mu+\nu, x+\mu+\nu+\alpha, \ldots, x+\mu+\nu+\alpha+\ldots+\sigma). \tag{5}$$

In order for the contour to be closed, $x = x+\mu+\nu+\alpha\ldots+\sigma$, one requires that each direction $\rho = 1, \ldots, d$ be traversed in opposite directions the same number of times and thus there are as many U_ρ's as $U_\rho^\dagger$'s in Eq. (4) for all $\rho = 1, \ldots, d$. In the following we shall demonstrate that in the large-N limit our Wilson loop amplitudes, Eq. (4), obey the same infinite set of identities, the so-called Schwinger-Dyson equations, as in the standard model under the identification Eq. (5). Hence if these identities uniquely specify the theory and in particular determine the values of Wilson loop amplitudes (which we assume to be the case), we conclude that $W_r(C)=W(C)$ and the theory, Eq. (1), is equivalent to its simpler version, Eq. (2).

Unfortunately our reduced model appears as yet too complicated to be solved exactly except for the extreme case of $d=2$ where we simply recover the results of Gross and Witten[3] and Wadia.[3]

Let us start our discussions by deriving Schwinger-Dyson equations in the U(N) gauge theory.[4] We consider an expression

$$\langle \mathrm{tr}(U_{x,\mu}\cdots U_{y-\nu,\nu}T^j U_{y,\alpha}U_{y+\alpha,\beta}\cdots U_{x-\sigma,\sigma})\rangle, \tag{6}$$

where T^j ($j=1,\ldots,N^2$) denotes a generator of the Lie algebra of U(N). By making an infinitesimal change of variable

$$U_{y,\alpha}\to(1+i\epsilon T^j)U_{y,\alpha} \tag{7}$$

and using the invariance of the integration measure we obtain

$$\begin{aligned}&\langle \mathrm{tr}(U_{x,\mu}\cdots U_{y-\nu,\nu}T^jT^jU_{y,\alpha}U_{y+\alpha,\beta}\cdots U_{x-\sigma,\sigma})\rangle\\ &+N/\lambda\langle (U_{x,\mu}\cdots U_{y-\nu,\nu}T^jU_{y,\alpha}U_{y+\alpha,\beta}\cdots U_{x-\sigma,\sigma})\mathrm{tr}(T^jU_{y,\alpha}U_{y+\alpha,\rho}U_{y+\rho,\alpha}{}^\dagger U_{y,\rho}{}^\dagger)\rangle\\ &-N/\lambda\langle \mathrm{tr}(U_{x,\mu}\cdots U_{y-\nu,\nu}T^jU_{y,\alpha}U_{y+\alpha,\beta}\cdots U_{x-\sigma,\sigma})\mathrm{tr}(U_{y,\rho}U_{y+\rho,\alpha}U_{y+\alpha,\rho}{}^\dagger U_{y,\alpha}{}^\dagger T^j)\rangle=0.\end{aligned} \tag{8}$$

After summing over j and making use of the formulas

$$\begin{aligned}&\sum_{j=1}^{N^2}(T^jT^j)_{ab}=N\delta_{ab},\\ &\sum_{j=1}^{N^2}(T^j)_{ab}(T^j)_{cd}=\delta_{ad}\delta_{bc},\end{aligned} \tag{9}$$

we arrive at the result

$$N\langle \mathrm{tr}\prod_{i\in C}U_i\rangle+\frac{N}{\lambda}\sum_{\rho\neq\alpha}\langle \mathrm{tr}\prod_{i\in C_\rho'}U_i\rangle-\frac{N}{\lambda}\sum_{\rho\neq\alpha}\langle \mathrm{tr}\prod_{i\in C_\rho''}U_i\rangle=0, \tag{10}$$

where contours are defined by

$$\begin{aligned}&C=(x,x+\mu,\ldots,y-\nu,y,y+\alpha,y+\alpha+\beta,\ldots,x-\sigma,x),\\ &C_\rho'=(x,x+\mu,\ldots,y-\nu,y,y+\alpha,y+\alpha+\rho,y+\rho,y,y+\alpha+\beta,\ldots,x-\sigma,x),\\ &C_\rho''=(x,x+\mu,\ldots,y-\nu,y,y+\rho,y+\rho+\alpha,y+\alpha,y+\alpha+\beta,\ldots,x-\sigma,x).\end{aligned} \tag{11}$$

When the link (y,α) occurs more than once in the original contour C, we obtain additional terms in Eqs. (8) and (10). For instance, in the case of a contour

$$C=(x,x+\mu,\ldots,y-\nu,y,y+\alpha,y+\alpha+\beta,\ldots,y-\gamma,y+\alpha,y+\alpha+\delta,\ldots,x-\sigma,x),$$

where the link (y,α) is traversed twice in the same direction, we obtain a term

$$-\langle(\mathrm{tr}\prod_{i\in C_1}U_i)(\mathrm{tr}\prod_{j\in C_2}U_j)\rangle \tag{12}$$

with

$$\begin{aligned}&C_1=(y,y+\alpha,y+\alpha+\beta,\ldots,y-\gamma,y),\\ &C_2=(y,y+\alpha,y+\alpha+\delta,\ldots,x-\sigma,x,x+\mu,\ldots,y-\nu,y),\end{aligned}$$

in the right-hand side of Eqs. (8) and (10). These terms correspond to δ functions in the right-hand side of Schwinger-Dyson equations in the continuum field theory and hence we call them source terms hereafter. We note that source terms always have the structure of the product of disconnected loops.

Now we discuss Schwinger-Dyson equations in the reduced model. When we repeat the derivation of these identities, we find identical equations except that there are now extra source terms which arise because some of the link variables are identified in our model. For instance, in the case of a contour

$$C=(x,x+\mu,\ldots,y-\nu,y,y+\alpha,y+\alpha+\beta,\ldots,z-\gamma,z,z+\alpha,z+\alpha+\delta,\ldots,x-\sigma,x)$$

there appear links (y,α) and (z,α). In the reduced model the same variable U_α is assigned to these links even for $y \neq z$. Thus we create an additional source term,

$$W(C_1, C_2) = \langle \mathrm{tr}(U_\alpha U_\beta \cdots U_\gamma)\mathrm{tr}(U_\alpha U_\delta \cdots U_\sigma U_\mu \cdots U_\nu)\rangle . \tag{13}$$

Here the sequences $(\alpha, \beta, \ldots, \gamma)$ and $(\alpha, \delta, \ldots, \sigma, \mu, \ldots, \nu)$ correspond to open paths

$$C_1 = (y, y+\alpha, y+\alpha+\beta, \ldots, z-\gamma, z)$$

and

$$C_2 = (z, z+\alpha, z+\alpha+\delta, \ldots, x-\sigma, x, x+\mu, \ldots, y-\nu, y),$$

respectively.

Thus the reduced model differs from the original one in general. In the limit of large N, however, we have the factorization property (the reduced model has only global gauge invariance; however, it is well known that the factorization holds also in matrix spin models in the large-N limit),

$$W(C_1, C_2) = \langle \mathrm{tr}\, U_\alpha U_\beta \cdots U_\gamma \rangle \langle \mathrm{tr}\, U_\alpha U_\delta \cdots U_\sigma U_\mu \cdots U_\nu \rangle + O(N^0) . \tag{14}$$

Now we notice that since C_1 and C_2 are open paths for $y \neq z$, there exists at least one direction ρ for which U_ρ and $U_\rho^\dagger$ appear different numbers of times in both of the sequences $(\alpha, \beta, \ldots, \gamma)$ and $(\alpha, \delta, \ldots, \sigma, \mu, \ldots, \nu)$. Then, making use of the fact that the measure and the action are invariant under the phase transformation $U_\rho \to e^{i\theta} U_\rho$, we find

$$\langle \mathrm{tr}\, U_\alpha U_\beta \cdots U_\gamma \rangle = \langle \mathrm{tr}\, U_\alpha U_\delta \cdots U_\sigma U_\mu \cdots U_\nu \rangle = 0 . \tag{15}$$

Therefore the contribution of the unwanted extra source terms is down by $1/N^2$ as compared with the other terms in the Schwinger-Dyson equations and hence may be ignored in the large-N limit [correct source terms as in Eq. (12) are, of course, kept intact]. Thus the Wilson loop amplitudes in the reduced model obey the same set of identities as in the standard model and must necessarily agree with the Wilson loop amplitudes of the standard theory.

The equality of the free energy of the reduced model, $F_r(\lambda)$, and the free energy per unit volume of the standard model, $F(\lambda)/V$, follows from the equality of the Wilson loop amplitudes. In fact the free energy is related to the internal energy $E(\lambda)$ as $\lambda^2/N\, dF(\lambda)/d\lambda = E(\lambda)$, and $E(\lambda)$ in turn is related to the Wilson loop amplitudes for elementary squares $W(C=1)$, $E(\lambda) = Vd(d-1)W(C=1)$. On the other hand $\lambda^2/N\, dF_r(\lambda)/d\lambda = E_r(\lambda) = d(d-1)W_r(C=1)$. Hence $F_r = F/V$ in the large-N limit.

It is easy to see that in the case of two dimensions our model reproduces the known results.[3] By introducing the δ function in the group space and its expansion in terms of characters, we have

$$Z_r = \int dU_1\, dU_2 \exp[\beta(\mathrm{tr}\, U_1 U_2 U_1^\dagger U_2^\dagger + \mathrm{tr}\, U_1 U_2^\dagger U_1^\dagger U_2)] = \int dU_1\, dU_2\, dV\, \delta(V, U_1 U_2 U_1^\dagger U_2^\dagger) \exp[\beta(\mathrm{tr}\, V + \mathrm{tr}\, V^\dagger)]$$
$$= \int dV \textstyle\sum_r [\chi_r(V)/d_r] \exp[\beta(\mathrm{tr}\, V + \mathrm{tr}\, V^\dagger)] . \tag{16}$$

This is to be compared with the partition function (per unit volume) of the standard theory,

$$Z = \int dV \exp[\beta(\mathrm{tr}\, V + \mathrm{tr}\, V^\dagger)] . \tag{17}$$

In Eq. (16) χ_r is the character of the irreducible representation r of the group U(N) and d_r denotes the dimensionality of the representation. The sum over r runs over all irreducible representations of $U(N)$. When we define $|r|$ to be the number of boxes in the Young tableau of the representation r, we have

$$d_r \approx N^{|r|}, \quad \chi_r(V) \approx (\mathrm{tr}\, V)^{|r|} \approx N^{|r|}, \tag{18}$$

and hence

$$\textstyle\sum_r [\chi_r(V)/d_r] \approx O(1) . \tag{19}$$

Therefore the free energy of the model, $F_r(\lambda) = -\ln Z_r/N^2$, agrees with that of the standard theory in the large-N limit.

In this paper we have pointed out an exciting possibility of a major reduction in the dynamical degrees of freedom in the large-N limit of lattice gauge theories. Our arguments were based upon the following assumptions: (1) unique specification of the theory by Schwinger-Dyson equations, and (2) factorization of disconnected amplitudes in the large-N limit. In the derivation of Eq. (15) we also implicitly assumed (3) the absence of spontaneous breakdown of the U(1) symmetry $U_\rho \to e^{i\theta} U_\rho$. These assumptions are known to be valid in perturbation theory and we expect them to hold for large values of λ. We have checked explicitly the agreement of the free energy of the standard and reduced models in lower orders of perturbation theory in arbitrary dimensions [up to $(1/\lambda)^5$]. It is extremely important to check if the equivalence of the models persists at small values of λ.

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Note added.—After the submission of this paper we received a preprint by G. Bhanot, U. Heller, and H. Neuberger where some evidence for a spontaneous breakdown of U(1) symmetry at small λ is presented. We understand that M. Peskin and K. Wilson have obtained similar results.

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Strangeness Production in the Quark-Gluon Plasma

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Rates are calculated for the processes $gg \to s\bar{s}$ and $u\bar{u}, d\bar{d} \to s\bar{s}$ in highly excited quark-gluon plasma. For temperature $T \geq 160$ MeV the strangeness abundance saturates during the lifetime ($\sim 10^{-23}$ sec) of the plasma created in high-energy nuclear collisions. The chemical equilibration time for gluons and light quarks is found to be less than 10^{-24} sec.

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Given the present knowledge about the interactions between constituents (quarks and gluons), it appears almost unavoidable that, at sufficiently high energy density caused by compression and/or excitation, the individual hadrons dissolve in a new phase consisting of almost-free quarks and gluons.[1] This quark-gluon plasma is a highly excited state of hadronic matter that occupies a volume large as compared with all characteristic length scales. Within this volume individual color charges exist and propagate in the same manner as they do inside elementary particles as described, e.g., within the Massachusetts Institute of Technology (MIT) bag model.[2]

It is generally agreed that the best way to create a quark-gluon plasma in the laboratory is with collisions of heavy nuclei at sufficiently high energy. We investigate the abundance of strangeness as function of the lifetime and excitation of the plasma state. This investigation was motivated by the observation that significant changes in relative and absolute abundance of strange particles, such as $\bar{\Lambda}$,[3] could serve as a probe for quark-gluon plasma formation. Another interesting signature may be the possible creation of exotic multistrange hadrons.[4] After identifying the strangeness-producing mechanisms we compute the relevant rates as functions of the energy density ("temperature") of the plasma state and compare them with those for light u and d quarks.

In lowest order in perturbative QCD $s\bar{s}$-quark pairs can be created by annihilation of light quark-antiquark pairs [Fig. 1(a)] and in collisions of two gluons [Fig. 1(b)]. The averaged total cross sections for these processes were calculated by

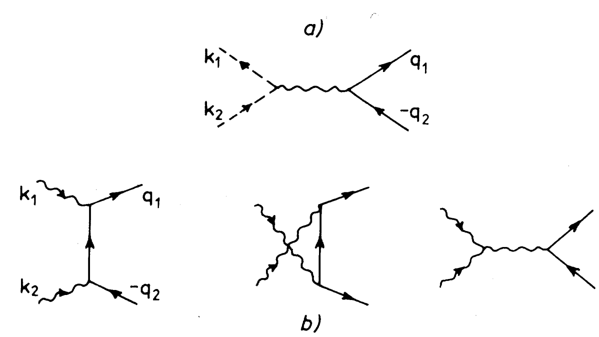

FIG. 1. Lowest-order QCD diagrams for $s\bar{s}$ production: (a) $q\bar{q} \to s\bar{s}$, (b) $gg \to s\bar{s}$.

