important observables in strongly deformed nuclei. The most conspicuous differences appear in the gap parameter, in the quadrupole moment, and in the single-particle occupation probabilities. The limitations implied by the *SD* subspace are thus not compatible with the coupling scheme appropriate for these nuclei. By extending the space to include pairs of particles coupled to 4 and 6, agreement is essentially obtained between the IBM and the Nilsson-plus-BCS model.

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[a]Permanent address: Comisión Nacional de Energiá Atómica, Buenos Aires, Argentina.

[b]On leave from Istituto Nazionale di Fisica Nucleare, Laboratori Nazionali Leguaro, Padova, Italy.

[c]Permanent address: Istituto di Fisica Galileo Galilei, Universitá di Padova, Padova, Italy, and Istituto Nazionale di Fisica Nucleare, Sezione di Padova, Italy.

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Equilibration in Finite Fermion Systems

Georg Wolschin

Max-Planck-Institut für Kernphysik, and Institut für Theoretische Physik der Universität, Heidelberg, Federal Republic of Germany

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A novel nonlinear transport equation for the time-dependent single-particle occupation numbers in an equilibrating fermion system is derived. In the case of constant transport coefficients its analytical solution together with an expression for the equilibration time is obtained. Applications in mean-field theories extended to include particle collisions for the description of low-energy heavy-ion reactions are envisaged.

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Dissipative heavy-ion collisions provide an ideal testing ground for equilibration processes in finite fermion systems. To account for both one-body and two-body dissipation[1,2] during the approach towards statistical equilibrium in a microscopic dynamical model, time-dependent Hartree-Fock calculations[3] have recently been supplemented with phenomenological collision terms.[4,5] They simulate the effect of two-body dissipation through a relaxation *Ansatz*. On the other hand, microscopic theories have been developed to derive collision terms or to investigate their structure.[6-9] They describe the time evolution of the single-particle occupation numbers in a given basis due to the residual interaction. The result of the microscopic theories has not yet been accessible for numerical calculations or analytical approximation schemes.

In this Letter I derive a nonlinear partial differential equation for the time-dependent single-particle occupation numbers in a finite fermion system. The transport coefficients occurring in the equation and thus the equilibration time are related to moments of the residual interaction. In the model case of constant transport coefficients I obtain the analytical solution through a nonlinear transformation. I expect this approach to be useful for the future understanding of dissipative nuclear collisions on the basis of extended mean-field calculations.

I start from the equation for the single-particle occupation numbers $n_\mu \equiv n(\epsilon_\mu, t)$, $0 \leq n_\mu \leq 1$, in a fermion system[6-8]:

$$\frac{\partial n_\mu}{\partial t} = \sum_{\alpha\beta\gamma} \langle V_{\mu\gamma\alpha\beta}{}^2 \rangle G(\epsilon_\mu + \epsilon_\gamma, \epsilon_\alpha + \epsilon_\beta)[(1-n_\mu)(1-n_\gamma)n_\alpha n_\beta - (1-n_\alpha)(1-n_\beta)n_\mu n_\gamma] \tag{1}$$

with the second moment $\langle V^2 \rangle$ of the residual interaction. In a finite system, the single-particle energies $\epsilon_\mu(t)$ as well as the basis states[8] $|\mu(t)\rangle$ depend on the mean field which changes in time in a nuclear collision. As a consequence, the energy-conserving function G is not a δ function as in the Boltzmann equation, but acquires a width of several megaelectronvolts. Its form has been derived in Ref. 8. Hence, collisions between nucleons in single-particle states that lie several megaelectronvolts apart become possible, and the action of the residual force is expected to yield a rapid equilibration in nuclear collisions.

Let us write the collision term in the form of a master equation with gain and loss terms, respectively:

$$\partial n_\mu / \partial t = (1-n_\mu) \textstyle\sum_\beta W_{\beta\to\mu} n_\beta - n_\mu \sum_\beta W_{\mu\to\beta}(1-n_\beta), \tag{2}$$

where

$$W_{\mu\to\beta} = \textstyle\sum_{\alpha\gamma} \langle V_{\mu\gamma\alpha\beta}{}^2 \rangle G(\epsilon_\mu + \epsilon_\gamma, \epsilon_\alpha + \epsilon_\beta)(1-n_\alpha) n_\gamma \tag{3}$$

and $W_{\beta\to\mu}$ accordingly. The aim is to transform the collision term into a tractable partial differential equation which contains the essential physical ingredients. The nonlinearity in the transition probability is presently disregarded; improvement with an iteration scheme is, however, conceivable. The β summation is replaced by an integration, thereby introducing the single-particle level density g_β to obtain

$$\partial n_\mu / \partial t = \int d\beta\, W_{\beta\mu}[g_\mu(1-n_\mu)n_\beta - g_\beta(1-n_\beta)n_\mu]. \tag{4}$$

I have defined $W_{\beta\to\mu} = W_{\beta\mu} g_\mu$, $W_{\mu\to\beta} = W_{\mu\beta} g_\beta$ and assumed $W_{\beta\mu} = W_{\mu\beta}$. Since G has a finite width, $W_{\beta\mu} = W(\frac{1}{2}(\epsilon_\beta + \epsilon_\mu), \epsilon_\beta - \epsilon_\mu)$ is peaked at zero and symmetric in $x = \epsilon_\beta - \epsilon_\mu$. This can be verified easily if one takes the equilibrium distribution for n_α and n_γ in Eq. (3) and $\langle V^2 \rangle = \text{const}$. An approximation to Eq. (4) is therefore obtained through a Taylor expansion of n_β and $g_\beta n_\beta$ around $\epsilon_\beta = \epsilon_\mu$. With the introduction of transport coefficients via moments of the transition probability

$$\begin{aligned} D &= \tfrac{1}{2} g_\mu \textstyle\int dx\, W(\mu, x) x^2, \\ v &= g_\mu{}^{-1} (d/d\epsilon_\mu)(g_\mu D), \end{aligned} \tag{5}$$

we arrive at the nonlinear partial differential equation for $n \equiv n(\epsilon_\mu, t)$,

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial \epsilon}\left[v\, n(1-n) + n^2 \frac{\partial D}{\partial \epsilon}\right] + \frac{\partial^2}{\partial \epsilon^2}[D\, n]. \tag{6}$$

This is the basic equation for the occupation-number distribution. The dissipative effects are expressed through the drift term v, the diffusive effects through the diffusion term D. Both transport coefficients combine to determine the speed of the relaxation process as well as the equilibrium distribution. Unlike in the case of linear diffusion equations of the Fokker-Planck type,[2] solutions of Eq. (6) are rather difficult to obtain. This is a consequence of the nonlinear terms which are due to the exclusion principle. They are essential for the proper description of the time evolution.

We consider this equation further in the limit of constant transport coefficients:

$$\frac{\partial n}{\partial t} = -v \frac{\partial}{\partial \epsilon}[n(1-n)] + D \frac{\partial^2 n}{\partial \epsilon^2}. \tag{7}$$

It has the correct Fermi-type equilibrium limit

$$n_\infty(\epsilon) = \{1 + \exp[-(v/D)(\epsilon - \epsilon_F)]\}^{-1}. \tag{8}$$

Here, the ratio of the two transport coefficients rather than a temperature determines the diffuseness. To treat the nonlinearity, we observe that

the transformation

$$n(\epsilon,t)=\frac{D}{vP(\epsilon,t)}\frac{\partial P(\epsilon,t)}{\partial\epsilon} \tag{9}$$

reduces Eq. (7) to a linear diffusion equation for $P(\epsilon,t)$,

$$P_t=-vP_\epsilon+DP_{\epsilon\epsilon}, \tag{10}$$

where $P_t\equiv\partial P/\partial t$, $P_\epsilon\equiv\partial P/\partial\epsilon$. Alternatively, we transform

$$w(\epsilon,t)=v-2vn(\epsilon,t), \tag{11}$$

and obtain for $w(\epsilon,t)$ the so-called Burgers equation[10]

$$w_t+ww_\epsilon=Dw_{\epsilon\epsilon}. \tag{12}$$

This equation has been used for the description of fluid flow and, in particular, shock waves in a viscous fluid.[10] The nonlinear transformation of Ref. 11,

$$w(\epsilon,t)=-2D\varphi_\epsilon/\varphi, \tag{13}$$

reduces it to the heat equation $\varphi_t=D\varphi_{\epsilon\epsilon}$. Hence, for a given initial distribution $n_0(\epsilon)$ of the single-particle occupation numbers, we solve Eq. (7) via Eqs. (9) and (10) or Eqs. (11)–(13) to obtain

$$n(\epsilon,t)=\int_{-\infty}^{+\infty}n_0(x)f(x,\epsilon;t)\,dx/\left[\int_{-\infty}^{+\infty}f(x,\epsilon;t)dx\right], \tag{14}$$

where

$$f(x,\epsilon;t)=\exp\{-\tfrac{1}{2}D^{-1}[vx-2v\int_0^x n_0(y)dy]\}\exp[-(\epsilon-x)^2/(4Dt)]. \tag{15}$$

I evaluate analytical solutions (solid curves in Figs. 1 and 2) for various initial distributions $n_0(\epsilon)$, dashed curves. In the lower part of Fig. 1,

$$n_0(\epsilon)=[1-\Theta(\epsilon-\epsilon_1)]+[1-\Theta(\epsilon-\epsilon_3)]\Theta(\epsilon-\epsilon_2) \tag{16}$$

is reminiscent of a light-ion–induced nuclear collision: The projectile particles initially occupy single-particle states in the continuum region of the potential, and the residual force then acts to equilibrate the system. Such a quasistatic picture is, however, oversimplified since it does not yet consider the time dependence of the mean field.

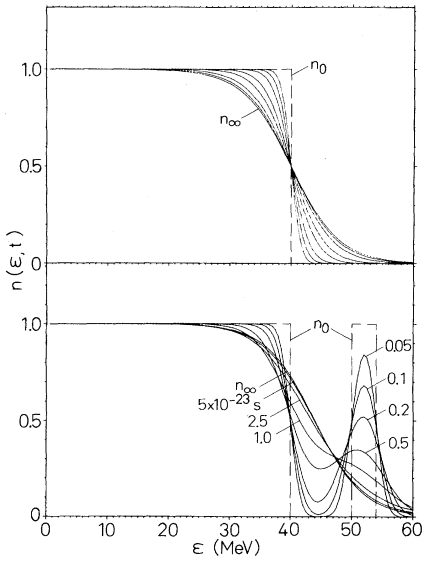

FIG. 1. Analytical solutions of the nonlinear differential equation (7) for the occupation-number distribution in a finite fermion system. The initial distributions (dashed curves) are n_0, the equilibrium distribution is n_∞. The transport coefficients are $D=20\times10^{23}$ MeV^2 s^{-1}, $v=-5\times10^{23}$ MeV s^{-1}. Times are in units of 10^{-23} s.

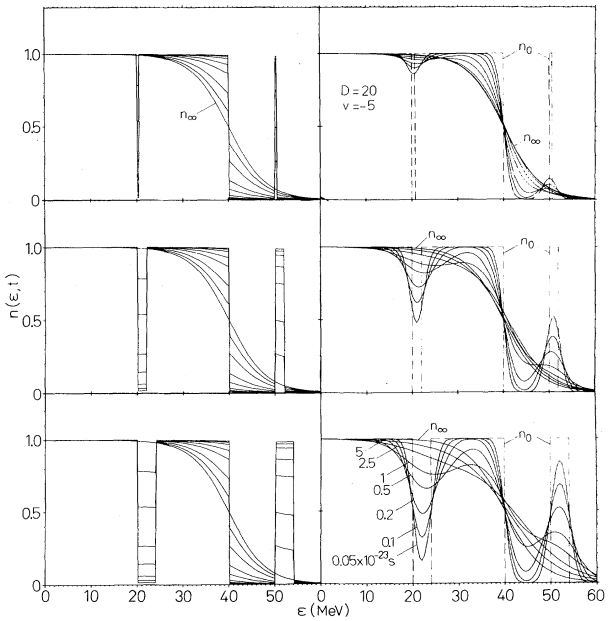

FIG. 2. Comparison of the results for the relaxation *Ansatz* (left-hand side, with $\tau_{equ}=4D/v^2$) and the analytical solutions of Eq. (7) (right-hand side). Three different initial distributions n_0 are shown. The relaxation *Ansatz* causes a slower equilibration at short times.

From the structure of the analytical solutions and the asymptotic values of the error function $\mathrm{erf}(\infty)=1$, $\mathrm{erf}(-\infty)=-1$ occurring in the solutions, it can be verified that they have the correct equilibrium limit (8), where the Fermi energy ϵ_F is given in terms of the initial values. For n_0 given by Eq. (16) we obtain $\epsilon_F=\epsilon_1-\epsilon_2+\epsilon_3$. This expresses particle-number conservation; cf. Fig. 1. From the asymptotic expansion of $\mathrm{erf}(z)$ it follows that the difference between actual and asymptotic solution vanishes like $\exp(-v^2t/4D)$. The equilibration time

$$\tau_{\mathrm{equ}}=4D/v^2 \tag{17}$$

is determined by the transport coefficients and thus, since $\tau_{\mathrm{equ}}\propto D/v^2\propto 1/\langle V^2\rangle$, essentially by the strength of the residual interaction that enforces the equilibration process. A microscopic calculation of τ_{equ} from Eqs. (5) and (17) is very desirable. Here, I choose $D=20\times 10^{23}$ MeV^2 s^{-1} and $v=-5\times 10^{23}$ MeV s^{-1} such that $-D/v=4$ MeV (to be compared with the temperature), and $\tau_{\mathrm{equ}}=3.2\times 10^{-23}$ s. The properties of the solutions are further explored in Fig. 2. Here, the initial distributions have hole regions of various extensions below and particle regions above ϵ_F. The solutions of Eq. (7) can still be obtained analytically. From the right-hand side of Fig. 2 it is apparent that the small disturbances are washed out much more rapidly than the bigger ones. On the left-hand side I compare with results for the corresponding relaxation *Ansatz* $n_t=(n_\infty-n)/\tau_{\mathrm{equ}}$ which enforces equilibration towards n_∞. The sequence of times is the same as on the right-hand side. For small times, and especially for small initial disturbances, the relaxation *Ansatz* leads to a much slower equilibration. This appears to be due to the fact that the nonlinearity is not treated explicitly. Hence, it seems worthwhile to replace the relaxation *Ansatz* by the solution of Eq. (7) in extended mean-field calculations.

To summarize, I have outlined a schematic model for equilibration in finite fermion systems. The master equation for the single-particle occupation numbers has been transformed into a nonlinear partial differential equation, keeping track of the exclusion principle in an essential way. I have derived the analytical solutions in the simplified case of constant transport coefficients. Dissipative and diffusive effects combine with the nonlinearity to yield the proper time evolution of the system towards the Fermi-type equilibrium limit. The equilibrium distribution will, however, be changed through the time dependence of the mean field. Hence, it seems to be of great interest to couple this model for the relaxation in the occupation numbers to the time evolution of the single-particle orbits[12] in order to get a complete description of the equilibration process.

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