Universal Fine Structure of the Chaotic Region in Period-Doubling Systems

T. Geisel and J. Nierwetberg

Institut für Theoretische Physik, Universität Regensburg, D-8400 Regensburg, Federal Republic of Germany

A new relation is reported which quantitatively describes the fine structure of the chaotic region of period-doubling systems. The relation determines the onset of fundamental periods and of ergodic behavior. It involves bifurcation rates γ_k, which converge to a new universal constant $\gamma = 2.948\,05\ldots$. This theory is in agreement with numerical determinations of γ_k.

Among the routes which lead to turbulent behavior the period-doubling route has attracted particular attention since Feigenbaum observed that there is quantitative universality.[1] This period-doubling route has been found in various physical systems. Libchaber and Maurer have observed it in a Rayleigh-Bénard experiment with small aspect ratio.[2] Among these systems there is also a class of driven anharmonic oscillators, e.g., driven Josephson junctions where the period-doubling route was found by Huberman and co-workers.[3,4] In typical experiments one measures spectra, i.e., Fourier-transformed time-correlation functions, where period doubling is manifested by subharmonic peaks. The existence of universal constants should be reflected in these experiments, e.g., in the intensity ratio of subharmonic peaks.[5]

Many properties of period-doubling systems appear to be explicable by simple iterative maps. Generally, when a parameter μ is varied a period 2^{k-1} becomes unstable and a period 2^k occurs at critical values μ_k which accumulate at μ_∞.[6] The sequence of these so-called *pitchfork* bifurcations is quantitatively described by Feigenbaum's relation $\mu_\infty - \mu_k \propto \delta^{-k}$, where δ is a universal constant.[1] While the period-doubling region of these systems is well understood, relatively little is known about the chaotic region, which sets in at μ_∞. Within the chaotic region there are periodic regimes where so-called *tangent* bifurcations[6] are responsible for the onset of new fundamental periods p (each of which is followed by a pitchfork cascade $p2^n$). It is known that certain periods must occur before others set in,[6,7] but quantitative statements about their onset are unknown. The purpose of the present Letter is to report a new (Feigenbaum-type) relation determining different sequences of tangent bifurcations within the chaotic region. These sequences are described by a new universal constant $\gamma = 2.948\,05\ldots$. The same relation also holds for parameter values with a certain chaotic behavior, which alternate with the onset of new periods. It thereby yields a more detailed picture for the succession and scaling of chaotic and periodic behavior within the chaotic regime. In experiments, the relation determines the emergence of new subharmonic spectral peaks of frequency $1/p$. We will report numerical calculations revealing this relation and describe a theory which yields γ in an approximate renormalization scheme.

We assume that the period-doubling systems mentioned above can be reduced to iterative maps of the form

$$x_{n+1} = f(x_n, \mu), \qquad (1)$$

where f is an analytic function of x_n with a single quadratic maximum.[7] An example is the logistic equation $x_{n+1} = \mu(x_n - x_n^2)$. The sequence of pitchfork bifurcations at μ_k accumulates at μ_∞ equal to 3.5699.... Above μ_∞ one finds a reverse sequence of bifurcations at parameters $\tilde{\mu}_k$ where 2^k chaotic bands merge into 2^{k-1} chaotic bands. Within a region of 2^k bands new periods p set in by a tangent bifurcation followed by a parameter window where p is stable. We have investigated sequences of periods $p = q2^k$ with k fixed and $q = 3$, $4, 5, \ldots$. The periods are defined by their itineraries. For the sake of simplicity we start by illustrating our results in Fig. 1 in the one-band region ($k = 0$). It shows the qth iterate $f^q(x_m, \mu)$ of the band maximum x_m. Whenever the qth iterate touches x_m there is a period q (which is superstable). There is a neighborhood where the period is stable. These cases are denoted by $P3$, $P4$, and $P5$ in Fig. 1, which illustrates that the sequence of periods $q = 3, 4, 5 \ldots$ accumulates at $\mu = \tilde{\mu}_0 = 4$. The cycles considered are characterized by an ascending order, $x_1 = f(x_m, \mu) < x_2 < \cdots < x_q$ $= x_m$, and are defined by itineraries[7] RLC, $RLLC$, $RLLLC$, $\ldots$. The cycles are also characterized as the last cycles of period q occurring before μ $= \tilde{\mu}_0 = 4$. These considerations can be generalized and the existence of such a sequence can be rigorously shown for a region of 2^k bands. In this sequence periods $p = q2^k$ (k fixed) set in at parameter values $\mu_{k,q}$ which accumulate at the band mergings $\tilde{\mu}_k$. Our main result is a recursion re-
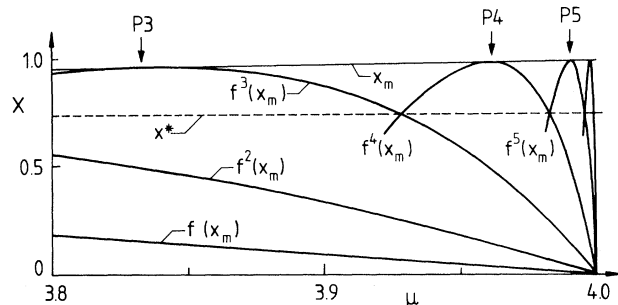
lation determining the onset of these periods $\tilde{\mu}_k$ $- \mu_{k,q} \propto \gamma_k^{-q}$. Here γ_k is the rate by which the distance from the accumulation point is scaled. γ_k converges to a new universal constant γ $= 2.94805 \ldots$. When we also include the prefactor the recursion relation[8] has the form

$$\tilde{\mu}_k - \mu_{k,q} = a\delta^{-k}\gamma_k^{-q}, \qquad (2)$$

where a is a constant and $\delta = 4.6692 \ldots$ is Feigenbaum's universal constant. This equation is based on numerical observations as well as on theoretical considerations, which will prove the universality of γ.

Before we explain Eq. (2) in more detail we note that there are still other sequences which follow the same recursion relation including the same constants γ_k: Above the band mergings ($\mu > \tilde{\mu}_k$) there are parameters where the $(q2^k)$th iterate of x_m falls on the (2^{k-1})th iterate. Thus there are sequences of periods $(2q - 1)2^{k-1}$ ($q = 2$, $3, 4, \ldots$) which accumulate at $\tilde{\mu}_k$ from above. They also have the same rate γ_k. Furthermore, related sequences exist for certain chaotic points: It is known that an ergodic invariant probability measure which is absolutely continuous exists if an iterate of x_m falls on an unstable cycle.[9] In Fig. 1 this happens where the $f^q(x_m, \mu)$ intersect the dashed line. These chaotic points also accumulate at $\mu = \tilde{\mu}_0 = 4$ and their distance scales at the same rate γ_0. Again a generalization to a region of 2^k bands is possible and a recursion relation like Eq. (2) holds for chaotic points where the $(q2^k)$th iterate of x_m intersects an unstable cycle of period 2^k. This happens on both sides of the band mergings $\tilde{\mu}_k$ and involves the same rates γ_k.

We will now present numerical tests of Eq. (2). The parameters $\mu_{k,q}$ belonging to periods $q2^k$ have been determined at the points of superstability of these periods, where highest accuracy can be achieved. The distance from the accumulation point $\tilde{\mu}_k$ is shown on a logarithmic scale in Fig. 2. In agreement with Eq. (2) the points lie close to straight lines even for small q. The slopes are determined by $-\ln\gamma_k$ and the vertical spacing of the lines is $\ln\delta$ as stated in Eq. (2). With increasing k the slopes approach a constant $-\ln\gamma$. We have obtained similar figures for other one-dimensional maps and have collected the values of γ_k in Table I. In the one-band and two-band region the values are still different but they approach the same value 2.9481 with a rapid convergence. This fact expresses the universality of the constant γ, which will be made more rigorous

FIG. 1. The band maximum x_m and its iterates as a function of μ calculated for the logistic map. $P3$, $P4$, and $P5$ denote superstable fundamental periods. At the intersections of $f^q(x_m)$ with the unstable fixed point x^* there is ergodic behavior.
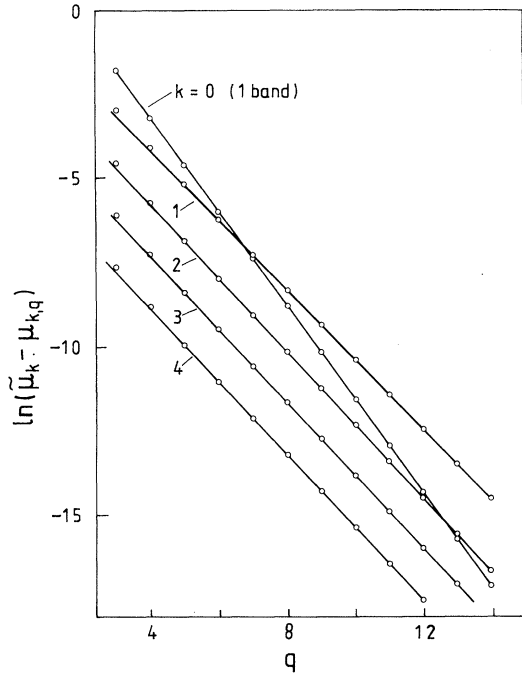
FIG. 2. Numerical test (logistic map) of the recursion relation Eq. (2) for superstable periods $q2^k$ for different numbers of bands 2^k. The slope of the lines approaches $-\ln\gamma$, and their spacing, $\ln\delta$.

TABLE I. Numerical and analytic values of γ_k in the 2^k-band region for various maps $f(x,\mu)$.

2^k	$\mu(x-x^2)$ numer.	$\mu(x-x^2)$ anal.	$xe^{\mu(1-x)}$ numer.	$\mu x(1-x^2)$ numer.
1	4.0000	4	$\cdots$	2.5981
2	2.8177	2.817 61	3.3605	2.9646
4	2.9634	2.963 32	2.8943	2.9472
8	2.9463	2.946 25	2.9549	2.9481
16	2.9483	2.948 27	2.9472	2.9481
32	2.9481	2.948 02	2.9482	2.9481

below.

In the following we investigate the parameter sequences theoretically and calculate γ_k and γ analytically. For convenience we do not carry out the theory for the sequence of new periods $q2^k$ but instead for the chaotic points where the iterate $f^{q2^k}(x_m, \mu)$ of the band maximum x_m intersects the unstable cycle of period 2^k (Fig. 1). As mentioned above the same recursion relation Eq. (2) accounts for these parameter values which will now be denoted by $\mu_{k,q}$. In the limit of large q the distance $\tilde{\mu}_k - \mu_{k,q}$ is determined by the inverse slope of $f^{q2^k}(x_m, \mu)$ as a function of μ:

$$\tilde{\mu}_k - \mu_{k,q} \propto \left[(d/d\mu)f^{q2^k}(x_m, \mu)\big|_{\tilde{\mu}_k} \right]^{-1}. \tag{3}$$

From Eq. (2) we expect this to be proportional to γ_k^{-q}. Introducing a new map

$$F(x, \mu) = f^{2^k}(x, \mu), \tag{4}$$

we can write $f^{q2^k}(x, \mu) = F^q(x, \mu)$. By mathematical induction it can be shown that

$$\frac{d}{d\mu}F^q(x, \mu) = \frac{dF}{d\mu} \prod_{\nu=1}^{q-1} \frac{\partial F}{\partial x}\bigg|_{F^\nu,\mu}$$

$$+ \sum_{l=1}^{q-1} \frac{\partial F}{\partial \mu}\bigg|_{F^l,\mu} \prod_{\nu=l+1}^{q-1} \frac{\partial F}{\partial x}\bigg|_{F^\nu,\mu}, \tag{5}$$

where we have dropped the arguments (x, μ). For $x = x_m$ and $\mu = \tilde{\mu}_k$ all the iterates $F^\nu(x_m, \tilde{\mu}_k)$ become equal to $F(x_m, \tilde{\mu}_k)$ and the products and sums can be carried out.[10] In the limit of large q this yields

$$\frac{d}{d\mu}F^q(x_m, \mu)\big|_{\tilde{\mu}_k} \propto \left[\frac{d}{dx}F(x, \mu)\big|_{x_{2^{k-1}}*,\tilde{\mu}_k} \right]^q, \tag{6}$$

where $x_{2^{k-1}}*$ is a point on the unstable cycle of period 2^{k-1} (or the trivial fixed point for $k = 0$). From Eqs. (3) and (4) it is clear that γ_k is given by the square brackets in Eq. (6):

$$\gamma_k = (d/dx)F(x, \mu)\big|_{x_{2^{k-1}}*,\mu_k}. \tag{7}$$

From this equation we can easily compute γ_k in the one-band case which for the logistic equation gives $\gamma_0 = \tilde{\mu}_k(1 - 2x_{tr}*) = 4$. Equation (7) represents an analytic expression for γ_k which can be carried out for any finite k if the band merging parameters $\tilde{\mu}_k$ are known. Using numerical values for $\tilde{\mu}_k$ we have calculated the analytical results for γ_k in the third column of Table I. These are more accurate than the numerical results in the second column. From Eq. (7) we can also determine the (universal) constant $\gamma = \lim\gamma_k$ with high accuracy (using a large k). We have thus obtained $\gamma = 2.948\,05\,\ldots$.

The universality of γ follows easily from Eq. (7). Assuming that $f(x, \mu)$ has its maximum at $x = 0$, we can introduce a topologically conjugate function $G = \lambda^{-k}f^{2^k}(\lambda^k x, \mu)$ instead of F, which leaves γ_k unchanged. In the limit $k \to \infty$ and $\mu = \mu_\infty$ the function G becomes a universal function[7] and therefore γ_k becomes a universal constant γ. One can also obtain an approximate analytic expression for γ in an approximate renormalization scheme.[10] Starting the renormalization at $k = 1$ in the logistic equation we obtain $\gamma = (2 - \mu_\infty)^2 = 2.4646$. Improvements are possible; e.g., starting the renormalization at $k = 2$ we obtain $\gamma = (-\mu_\infty^2 + 2\mu_\infty + 4)^2 = 2.5741$.

We resume with what we now know about the fine structure of the chaotic regime. Above and below the band mergings $\tilde{\mu}_k$ sequences of tangent bifurcations of period $q2^k$ and $(2q-1)2^{k-1}$ accumulate at $\tilde{\mu}_k$. The parameters $\tilde{\mu}_k$ are not only accumulation points for stable periods but also for ergodic behavior on both sides of $\tilde{\mu}_k$. These accumulations all follow the rate γ_k which becomes universal for large k. The accumulation points $\tilde{\mu}_k$ by themselves accumulate at μ_∞ at a rate δ. On a finer scale each tangent bifurcation (e.g., $q2^k$) is followed by a subharmonic cascade (e.g., $q2^k2^n$ with q and k fixed). This cascade accumulates again with the rate δ. This quantitative fine structure should become observable in experiments where spectra are measured, e.g., in a Rayleigh-Bénard fluid or in a Josephson junction. In a Rayleigh-Bénard experiment the role of the parameter μ is played, e.g., by the Rayleigh number. The onset of a new period p is marked in the spectrum by the occurrence of a subharmonic peak at frequency $1/p$. The critical values of the Rayleigh number where this happens should obey our recursion relation Eq. (2) (e.g., for a sequence of fundamental periods $3, 4, 5 \ldots$). It has been found recently that the presence of external fluctuations may inhibit large periods depending on the width of the parameter windows where they are stable.[11-13] The experiments are thus required to be relatively free from external noise. Libchaber and Maurer[2] have observed subharmonics belonging to period 16. From the width of the window of period 16 we can estimate that, e.g., the sequence of fundamental periods $3, 4, 5, \ldots$ should be observable up to period 5 under the same circumstances.

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Note added.—(a) The widths of parameter windows are also scaled down by γ. Here a window may be understood as a parameter interval for which a certain period is stable or to which a cascade $p2^n$ extends. An equation similar to Eq. (2) holds for the widths. Therefore, if the width of a particular window (e.g., of period 3) is known one can estimate the width for any other fundamental period mentioned in this paper and also for their subharmonics. Denoting by $\Delta\mu_{k,q,n}$ the width of the nth subharmonic of a fundamental period $q2^k$ the estimated width is $\Delta\mu_{k,q,n} = \Delta\mu_{0,3,0} \times \delta^{-k-n}\gamma^{3-q}$. (b) Equation (2) also determines the onset of intermittent chaos and it seems likely that the widths of windows of intermittency are also scaled by γ.

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