Time-Dependent Order Parameters in Spin-Glasses

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It is proposed that the spin-glass phase is described by two order parameters $q(x) = [\langle S_i(0)S_i(t_x)\rangle]_{av}$ and $\chi(x) = \int_0^{t_x}\chi(t)\,dt$, that measure the relaxations of the average autocorrelation and susceptibility along *macroscopic* time scales, t_x, which are parametrized in a *decreasing* order by $x\in[0,1]$. The function q decays from a finite value at t_1 to zero at t_0, while χ increases from $\chi(1)$ to a value $\chi(0)$ which is independent of temperature in the mean-field limit. The equilibrium results agree with Parisi's replica solution.

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The spin-glass (SG) phase in magnetic systems with random exchange is usually characterized[1] by the Edwards-Anderson[2] (EA) order parameter $q_{EA}=[\langle S_i\rangle^2]$ (where $\langle\ldots\rangle$ and $[\ldots]$ mean thermal average and random average, respectively) which describes freezing of local spins S_i in random directions. Although it is by now clear that q_{EA} is not a sufficient order parameter,[3] the nature of the additional SG order parameters has been an open question. Recently, using a particular replica ($n\to 0$) symmetry breaking scheme, Parisi[4] introduced as a SG order parameter a function $q(x)$, $x\in[0,1]$. Although Parisi's mean-field solution has many desired properties, several questions remain unanswered. Most important, the physical meaning of the replica broken symmetry, as well as that of the order parameter $q(x)$, is completely mysterious. Also, calculation of some physical quantities, such as q_{EA}, is ambiguous.[4,5] Finally, the solution was found[5] to be locally unstable unless one restricts the space of allowed functions $q(x)$.

In this note, we investigate the static properties of the SG phase using a dynamic approach previously developed[6] for the SG problem. We work explicitly within the infinite-ranged Ising SG model,[7] given by the Hamiltonaian $\sum_{ij}J_{ij}S_iS_j$ where each J_{ij} is randomly distributed around zero, in which a proper mean-field theory is exact. I believe, however, that the qualitative features of the theory apply also to finite-ranged systems whenever a SG transition actually occurs. The true equilibrium properties of the mean-field solution agree with Parisi's[4] results near T_c and most probably at all temperatures as well. By incorporating explicitly the slow time relaxation of the order parameters I was able to construct, for the first time, a consistent mean-field theory which takes account of both the true equilibrium and the nonequilibrium phenomena associated with the SG order. I first describe the results.

The dynamic analog of q_{EA} is the time-persistent spin-spin correlations $q_{EA}=\lim_{t\to\infty}[\langle S_i(0)\times S_i(t)\rangle]$. It was noted earlier[6] that the thermodynamic limit ($N\to\infty$) is not uniquely defined unless one incorporates also the time decay of q_{EA} in finite N. Here I assume that this decay does not occur in a single (macroscopic) relaxation time but rather in a distribution of many large relaxation times, t_x, *all of which become infinite in the thermodynamic limit*, which I parametrize in a *decreasing* order by an arbitrary parameter $0\le x\le 1$. The time t_0 is the longest time scale (i.e., the purely static limit) whereas t_1 is the shortest one, which approaches the *finite*-time spectrum of the system. In general, if $x>x'$, then I assume $t_{x'}/t_x\to\infty$ as $N\to\infty$. This leads to a generalized order parameter

$$q(x)=[\langle S_i(0)S_i(t_x)\rangle], \tag{1}$$

which measures the amount of correlations which have not decayed at the time scale t_x. Thus, $q(x)$ is a monotonic increasing function with a maximum value $q(1)=q_{EA}$ which is the frozen correlations measured in *finite* time. Another order parameter is $\Delta(x)=T\chi(x)-(1-q_{EA})$, where $\chi(x)$ is the local susceptibility measured at the frequency $\omega_x=t_x^{-1}$, i.e.,

$$\chi(x)=\mathrm{Re}\,\chi(\omega_x)=\int_0^{t_x}\chi(t)\,dt. \tag{2}$$

The order parameter $\Delta(x)$ is the slow response due to overturning of large clusters, and hence is a decreasing function of x with its maximum value $\Delta(0)$ in the purely static susceptibility $\chi(0)$.

The functions $q(x)$ and $\Delta(x)$ are assumed to be continuous. This underlines the crucial physical assumption that the order parameters are sums of a large number of contributions from a broad continuum of time scales ranging from the extreme static to the finite-time limits. This implies also that

$$q(x)\to 0 \text{ as } x\to 0 \tag{3}$$

in zero field, manifesting the complete decay of frozen correlations at the largest time scale, and,

$$\Delta(x) \to 0 \text{ as } x \to 1, \quad (4)$$

reflecting the validity of linear-response theory in the dynamic region.

Using the dynamic theory developed in Ref. 6, I have derived self-consistent equations for $q(x)$ and $\Delta(x)$ in the case of infinite-ranged Ising spin-glass. The above assumptions were found to be self-consistent at all temperatures below T_c and furthermore lead to a solution with many unique properties. In particular, the existence of a critical order parameter $q(x) \sim 0$ as $x \sim 0$ leads directly to a constant static (zero field) susceptibility

$$\tilde{J}\chi(0) = 1 \quad (5)$$

at all temperatures below T_c. The criticality of the SG phase is also manifested in the spectrum of the staggered susceptibility $\chi_\lambda = \partial\langle S_\lambda\rangle/\partial h_\lambda$ associated with eigenstates $|\lambda\rangle$ of the random exchange matrix J_{ij}.[8] I find that $\chi_\lambda^{-1}(x) = r(x) + \beta(J_{\max} - J_\lambda)$ (J_λ being the eigenvalue of the state $|\lambda\rangle$), where $r(x)$ exhibits a crossover from a finite positive value at short times ($x \sim 1$) to $r(x) \to 0$ for $x \to 0$, implying the divergence at all $T \leqslant T_c$ of the static "SG susceptibility"[5] as well. Similarly, a consequence of Eq. (4) is that there is no gap in the relaxational spectrum of the finite-frequency susceptibility. In fact, Eq. (4) leads to the relation

$$T^2/\tilde{J}^2 = 1 - 2q_{\rm EA} + [\langle S_i\rangle^4], \quad (6)$$

from which it follows[6] that the finite-time autocorrelations decay algebraically $\sim t^{-\nu}$ with $\nu \leqslant \frac{1}{2}$. (Note that $[\langle S_i\rangle^4]$, like $q_{\rm EA}$, refers to finite-time value of frozen spins.) Expansion near T_c to $O(\tau^3, h^2)$ ($\tau \equiv 1 - T/T_c$, h is a static field) yields

$$q_{\rm EA} = q(1) \simeq \tau + \tau^2 - \tau^3, \quad (7)$$

$$\tilde{J}\chi(0) \simeq 1 - (3h^2/4)^{2/3}(1 - 2\tau/3), \quad (8)$$

$$(1 + 3\tau)\Delta(x) + q^2(x) \simeq \tau^2 + 2\tau^3, \quad (9)$$

where $\tilde{J} \equiv N[J_{ij}^2]$. Note that since the scale of x is arbitrary, the theory can only establish a relation between $q(x)$ and $\Delta(x)$ but cannot determine *both* of the functions, except at the uniquely defined end points 0 and 1.

The theory thus predicts two distinct features for the time-dependent susceptibility. In a time scale which is large compared to microscopic processes but small with respect to macroscopic ones the susceptibility approaches a quasiequilibrium value $\chi(1) = \beta(1 - q_{\rm EA})$ ($\beta \equiv 1/k_BT$) with a power-law behavior $\sim t^{-\nu-1}$. Then the susceptibility further relaxes to its true equilibrium value $\chi(0) = \beta[1 - q_{\rm EA} + \Delta(0)]$; this, however, is expected to occur on a much slower rate spanning a broad range of macroscopic time scales. The order parameter Δ provides a much clearer definition of the SG transition than q, especially since many experiments are done in a finite field. In the presence of a field, the theory predicts the existence of a critical line $T_c(h)$ for Δ, although q is nonzero everywhere. This line, which was previously derived[3-5] in the context of replica symmetry breaking, marks the vanishing of Δ and hence the complete disappearance of the irreversible phenomena at $T > T_c(h)$. At present, we can estimate neither the time scale nor the rate at which the slow relaxation occurs, and thus even the measurability of this relaxation may be questioned. On the other hand, in three-dimensional short-ranged systems, these long relaxation times may be finite even in the thermodynamic limit. Indeed, low-field magnetic measurements[9,10] in spin-glasses do exhibit a slow relaxation of the susceptibility from a nonequilibrium value χ_1 towards an equilibrium one χ_0. The qualitative features of these susceptibilities suggest the identification of χ_0 and χ_1 with our susceptibilities $\chi(0)$ and $\chi(1)$, which then provides a simple way of measuring both $q(1)$ and $\Delta(0)$ as a function of T. Finally, it is important to note that the predicted relaxation of χ to the value $\chi(0) = \tilde{J}^{-1}$ occurs on the same time scale in which the autocorrelations decay completely to zero. This suggests that studies of the onset of SG transition by computer simulations[1] should concentrate on the time dependence of the actual response function rather than on that of the EA order parameter. I proceed to outline the derivation of these results.

Relaxational dynamics is introduced[6] for the infinite-ranged SG model[7] together with a local Gaussian noise φ which simulates the approach to equilibrium. We write the average dynamic autocorrelation and local response as

$$C(\omega) = [\langle S_i(\omega)S_i(-\omega)\rangle_\varphi] = \tilde{C}(\omega) + q\,\delta(\omega),$$

$$\chi(\omega) = [\partial\langle S_i(\omega)\rangle_\varphi/\partial h_i(\omega)] = \tilde{\chi}(\omega) + \Delta\delta_{\omega,0},$$

where $\tilde{C}$ and $\tilde{\chi}$ are the finite-frequency parts of C and χ, related through $\tilde{C}(\omega) = 2\omega^{-1}\,{\rm Im}\tilde{\chi}(\omega)$ and $\langle\ldots\rangle_\varphi$ means average with respect to the noise φ. The symbols $\delta_{\omega,0}$ and $\delta(\omega)$ are, respectively, Kronecker and Dirac δ functions. In the case of infinite-ranged exchange J_{ij}, the random average is done by using a self-consistent Gaussian noise which is a sum of a "fast" noise f and time-persistent noise z, $\langle z(\omega)z(-\omega)\rangle = q\delta(\omega)$. In Ref. 6 it was shown that the order-parameter q is given as

$$q\delta(\omega) = \langle\langle S_i\rangle_f^2\rangle_z, \tag{10}$$

where $\langle S_i\rangle_f$ is the equilibrium magnetization induced by a time-persistent effective field H,

$$H\{z\} = \beta\tilde{J}z(\omega) + \beta^2\tilde{J}^2\Delta\delta_{\omega,0}\langle S\rangle_f + \beta h. \tag{11}$$

The order parameter Δ is given by

$$\chi = \beta(1 - q + \Delta) = \partial\langle\langle S\rangle_f\rangle_z/\partial h. \tag{12}$$

It has been noted[6] that because of the appearance of ill-defined products $\Delta\delta_{\omega,0}q\delta(\omega)$, the solution of Eqs. (10)–(12) depends, even in the limit $N \to \infty$, on the frequency dependence of q and Δ in finite N. Instead of solving a dynamic problem in finite N, we proceed by making certain assumptions about the low-frequency singular structure of C and χ. The motivation for this is the physical requirement that the static solution in the $N \to \infty$ limit be independent of any time scale as well as any other details of a particular dynamic model. To this end, let us envisage a phase space with many ground states separated by macroscopic energy barriers b_i. This gives rise to a distribution of relaxation times $t_i \sim \exp(b_i/T)$ such that for any pair $b_i < b_j$, $t_i/t_j \ll 1$. Consequently, we write

$$q\delta(\omega) = \sum_{i=1}^{k} q_i'\delta^i(\omega),$$

$$\Delta\delta_{\omega,0} = -\sum_{i=1}^{k}\Delta_i'\delta_{\omega,0}{}^i, \tag{13}$$

where $\delta^i(\omega)$ and $\delta_{\omega,0}{}^i$ are, in the limit $N\to\infty$, normalized Dirac δ functions and Kronecker δ but for finite N have characteristic widths $\omega_i \sim t_i^{-1}$ which are parametrized in an increasing order (hence q_i', $-\Delta_i' \geq 0$). Similarly, we write the total noise z as $z = \sum_i^k z_i$ with $\langle z_i(\omega)z_i(-\omega)\rangle = q_i'\delta^i(\omega)$. Since $\omega_i/\omega_j \ll 1$ for every $i<j$, $\delta^i(\omega) \times \delta_{\omega,0}{}^j = \delta^i(\omega)$ if $i<j$ and 0 if $i>j$. This together with an arbitrary definition of $\delta^i(\omega)\delta_{\omega,0}{}^i$ defines for any number of modes, k, a unique self-consistent solution of Eqs. (10)–(12). This sequence of solutions has a well-defined limit as $k\to\infty$. This limiting solution is the one which is adopted here, firstly because physically one expects that in the limit $N\to\infty$ there is a continuous distribution of barriers b_i and hence also of t_i. Also, in this limit ($N, k\to\infty$) the contribution q_i' and Δ_i' from each time scale is infinitesimal, and hence the detailed dynamic properties of $\delta^i(\omega)$ and $\delta_{\omega,0}{}^i$ within each "decade" [e.g., the value of $\delta^i(\omega)\delta_{\omega,0}{}^i$] become irrelevant. Defining a continuous parameter $x \equiv i/k$, we construct the partial sums $q(x) = \int_0^x q'(y)\,dy$, $\Delta(x) = -\int_x^1 \Delta'(y)\,dy$ [$dx \equiv 1/k$, $f'(x) \equiv kf'(i)$] which are equivalent to Eqs. (1) and (2). In terms of these functions, Eq. (11) reads

$$H\{z\} = \beta\tilde{J}\int_0^1 z(x)\,dx - \beta^2\tilde{J}^2\int_0^1 \Delta'(x)m_x\{z\}dx + \beta h, \tag{14}$$

where $z(x)$ has a Gaussian measure $\langle z(x)z(x')\rangle = \delta(x-x')q'(x)$ and m_x is the value of the local magnetization which remains frozen at the time scale t_x, i.e.,

$$m_x\{z\} = \int_{-\infty}^{\infty}\prod_{y>x}\left(\frac{dz(y)}{[2\pi q'(y)]^{1/2}}\right)\exp\left[-\frac{1}{2}\int_x^1 dy\,\frac{z^2(y)}{q'(y)}\right]\tanh(H\{z\}). \tag{15}$$

Equations (10) and (12) are now

$$q(x) = \langle m_x^2\rangle_z, \tag{16}$$

$$\tilde{J}\chi(x) = \tilde{J}\beta[1 - q(1) + \Delta(x)] = \langle\delta m_1\{z\}/\delta z(x)\rangle_z[q'(x)]^{-1/2}, \tag{17}$$

where the last equality is derived by adding to H an infinitesimal field h_x with a characteristic frequency ω_x. Expansion of Eqs. (15)–(17) near T_c yields the results (7)–(9). In addition, expanding the right-hand side of Eq. (16) at all $T \leq T_c$ for $x \sim 0$ yields $q(x) \simeq [\tilde{J}\chi(0)]^2 q(x) + O(q(x)^2)$ from which Eq. (5) readily follows. Similarly, Eq. (6) follows by expanding the right-hand side of Eq. (17) for $x \sim 1$ which yields

$$\Delta(x) \simeq \beta^2\tilde{J}^2[1 - 2q(1) + \langle m_1^4\rangle_z]\Delta(x) + O(\Delta(x)^2).$$

Other thermodynamic quantities may be derived with use of the original dynamic Lagrangian.[6] Alternatively, we use the above theory to generalize previous derivations of Sommers's free energy,[8,11,12] yielding a free-energy functional

$$-\beta F\{q,\Delta\}=\tfrac{1}{4}\beta^2\tilde{J}^2\{[1-q(1)]^2+2\int_0^1\Delta'(x)q(x)\,dx\}$$
$$+\int_{-\infty}^{\infty}\prod_x\left(\frac{dz(x)}{[2\pi q'(x)]^{1/2}}\right)\exp\left(-\frac{1}{2}\int_0^1\frac{z^2(x)}{q'(x)\,dx}\right)[\ln 2\cosh(H\{z\})+\tfrac{1}{2}\beta^2\tilde{J}^2\int_0^1\Delta'(x)m_x^{\,2}\{z\}dx] \quad (18)$$

[with H and m_x defined in Eqs. (14) and (15)] which reproduces Eqs. (16) and (17) via $\delta F/\delta q'(x) = \delta F/\delta\Delta'(x)=0$. As pointed out in the beginning, the theory determines in the region $0<x<1$ only a relation between $q(x)$ and $\Delta(x)$. By eliminating one of these quantities one can define the theory in terms of a single unique function of x. For instance, Parisi's solution $q(x)$ corresponds to $\Delta'(x)=-xq'(x)$. In that case, however, the physical range of x is from $x_0=\min\{|\Delta'(x)|/q'(x)\}$ to $x_1=\max\{|\Delta'(x)|/q'(x)\}$. Thus the flat regions (x between 0 and x_0 or between x_1 and 1) which were the origin of the apparent instability[5] of Parisi's solution are a formal artifact of his particular replica scheme and do not exist in the present theory. Finally, it is noted that a theory in terms of only one function is justified only in the mean-field limit, i.e., at the stationary value of Eq. (18). Once fluctuations are introduced there is no reason why *independent* variations in $q(x)$ and $\Delta(x)$ should not be considered. Allowing for such fluctuations complicates the stability analysis of Eq. (18) and requires further study.

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