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Pattern Selection in Rayleigh-Bénard Convection near Threshold

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A correct long-wavelength theory for convection near onset with free-slip boundary conditions requires two fields and reversible couplings. The wavelengths for which stable rolls exist are dramatically modified when the generation of vertical vorticity is taken into consideration. For small Prandtl numbers and rigid boundaries, the skewed-varicose instability of Busse and Clever is recovered by a plausible but nonrigorous modification of our free-slip equations.

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Among the many examples of symmetry breaking and pattern formation in nonequilibrium systems, Boussinesq convection has probably been subject to the most intense experimental and theoretical scrutiny.[1] Because the bifurcation is a continuous one, only a slow modulation of the basic convective roll pattern is allowed by the fluid equations near onset. The time evolution of a general pattern is naturally developed by means of multiscale perturbation theory in terms of a complex amplitude, A, whose phase describes changes in the position and direction of the rolls and whose magnitude modulates the intensity of the convective motion. The dynamical equation for A concisely resolves the question of which of the highly degenerate manifold of linearized solutions to the Boussinesq equations persist and are locally stable when nonlinear effects are included.[2-4] Prior to the work reported here, this equation, to lowest nontrivial order in the deviation of the Rayleigh number or bifurcation parameter, R, from its critical value, R_c, was thought to be purely relaxation, i.e.,

$$dA/dT = -\delta F/\delta A,$$

where F is a local functional of A.[2] As such, it would also settle the question of global stability for the convection problem near onset, since if a small amount of noise were introduced so as to allow the system to sample the manifold of locally stable states accessible to it, the roll pattern would evolve toward the one state that minimizes the potential F.

It therefore came as a surprise when we found that the relaxational equation for A is seriously in error for free-slip boundary conditions when the Prandtl number, P, is finite. The correct equations contain a convective term, which invalidates any statement that could formally have been made about global stability. Furthermore, the band of locally stable wave numbers near onset changes dramatically and for $P \lesssim 10.0$, it is effectively the mirror image of what was previously considered correct.

For the rigid boundary conditions appropriate to experiment, the relaxational equation for dA/dT is correct to lowest order in $R-R_c$ but its domain of validity is severely limited as $P\to 0$. We propose a simple but nonrigorous modification of our free-slip equations that eliminates the one unphysical feature of these boundary conditions, and captures what we believe to be the most important nonrelaxational corrections to dA/dT for rigid boundaries and small P. In fact for $P\lesssim O(1)$, there is a natural continuity between free-slip and rigid boundaries provided $R-R_c$ is not too small. The new instabilities we found under the former conditions pass over to the "skewed-varicose" instability of Busse and Clever.[5] Our model for rigid boundaries may be of some relevance to the experimental observations of persistent low-frequency noise in large containers at low Prandtl numbers near onset, a finding incompatible with purely relaxational dynamics.[6]

We finally consider how rigid lateral boundaries influence the stationary states accessible to the bulk. We consider a model problem with spatial variation only along the roll axis and argue that for rolls constrained to be normal to two slowly diverging rigid walls and otherwise smooth, no steady states exist.

Let us define in terms of the critical value of R an expansion parameter $\epsilon=(R-R_c)/R_c$, and introduce length and time scales, $\xi_x^{-2}=3\pi^2\epsilon/8$, $\xi_y^{2}=\xi_x/(\sqrt{2}\,\pi)$, and $\tau^{-1}=1.5\pi^2\epsilon P/(1+P)$.[3] Unscaled variables or fields (X) will be denoted by $\tilde{X}$ and we assume that the unperturbed roll pattern is parallel to the y axis and has a wavelength $2\pi/q_0$. To lowest nontrivial order in ϵ we find

$$\partial_t A = A+(\partial_x - i\,\partial_y{}^2)^2A-|A|^2A-iB_xA, \tag{1a}$$

$$\gamma\,\partial_t\Omega_z=\partial_y{}^2\Omega_z+g\,\partial_y[A^*(\partial_x-i\,\partial_y{}^2)A+\text{c.c.}], \tag{1b}$$

where $\gamma=\sqrt{3}\,\epsilon^{1/2}/(1+P)$, $g=2(1+P)/P^2$, and c.c. denotes the complex conjugate. Equation (1b) is new and describes the generation of vertical vorticity, Ω_z, from the roll curvature. With our scalings the slowly varying piece of the x velocity in physical units, $\tilde{B}_x$, is recovered from

$$\Omega_z=-\partial_yB_x,\quad B_x=2q_0(1+P)\tilde{B}_x/(3\pi^2P\epsilon). \tag{2}$$

By contrast, A gives rise to a contribution to the x velocity of the form $c\,\mathrm{Re}[\exp(iq_0\tilde{x})A]\cos(\pi z)$, where $c=O(\epsilon^{1/2})$. Although free-slip boundary conditions are peculiar in allowing a horizontal velocity independent of z,[7] vertical vorticity is always present, though to higher order in ϵ whenever rolls bend.

The other component of lateral velocity, B_y, is found from Ω_z and B_x by imposing incompressibility. Neglecting terms that could only originate from the initial conditions and are not generated by (1b) implies that $\tilde{B}_y\sim O(\epsilon^{5/4})$ and that it only enters (1a) to next order in ϵ in the form $\epsilon^{1/2}B_y\times\partial_yA$. There is no obvious problem with (1b) near rigid lateral boundaries since the source of vorticity vanishes with $|A|$. The variation of a defect's velocity with Prandtl number that we have observed[8] is clearly consistent with (1b) since $\Omega_z\sim O(1/P)$ for $P\gg 1$; moreover, a quantitative calculation of the effect should be possible. We note in passing that (1a) with $B_x=0$ is only correct when the rolls remain rectilinear or if $P=\infty$. The problem of a cylindrically symmetric roll pattern needs to be reexamined.[9]

The different scale factors applied to x and y were chosen to obtain the conventional form of (1a),[3] but are clearly inappropriate for (1b). When we present the stability analysis for the laterally periodic problem it will be necessary to go from Ω_z to B_x by using the full two-dimensional Laplacian to avoid an unphysical nonuniformity in the limits $\epsilon\to 0$ and (box size) $\to\infty$.

While it is clear from (1a) and (1b) that A is the slowest mode (i.e., the order parameter) and Ω_z is faster and derived from it, the limit $P\to 0$ is singular. Indeed, Busse has shown that the oscillatory instability, for which Ω_z is the order parameter, occurs for $\epsilon^{1/2}\sim P$ in an infinite system.[7] If we set

$$A=1+e^{-\lambda t}(\delta A e^{iky-i\omega t}-\text{c.c.}),\qquad B=\delta B e^{-\lambda t}\cos(ky-\omega t), \tag{3}$$

in (1) and (2) and linearize in δA and δB, we obtain to the order in which we work precisely the eigenvalues and eigenmodes predicted by Busse[7] and later used by McLaughlin and Martin.[10] We thus infer that (1a) and (1b) are limited to $\epsilon\ll\min(1,P^2)$ and that higher-order terms in ϵ/P^2 will reverse the sign of the dissipation in (1b).[11,12]

To investigate the stability of parallel rolls in a large but finite, laterally periodic container we set $A=(1-q^2)^{1/2}e^{i(qx+\varphi)}(1+u)$ and linearize in u, B_x, and the gradients of φ. The condition that the perturbation with wave vector (k_x,k_y) be unstable is

$$2g[qk_x{}^2-(1-q^2)k_y{}^2]/(1+\delta k_x{}^2/k_y{}^2) >(k_y{}^2+\delta k_x{}^2)(\alpha k_x{}^2+2qk_y{}^2+k_y{}^4), \tag{4}$$

where $\delta = \xi_y^2/\xi_x^2$ and $\alpha = (1-3q^2)/(1-q^2)$. It is then obvious on dimensional grounds that for $q > 0$ parallel rolls are always unstable in a sufficiently large box whose minimum dimensions tend to infinity with P. The denominator on the left-hand side of (4) arises from the replacement, $\Omega_z = -ik_y B_x(1+\delta k_x^2/k_y^2)$, mentioned above which eliminates the unphysical instability at $k_y = 0$ for $q > 0$.

For $q < 0$, instability requires a finite k_y and we can therefore set $\delta = 0$ in (4) and investigate only $k_x = 0$. The so-called zigzag instability first occurs at $k_y^2 = -q$, provided

$$q^2/(1-q^2) \geq 4(1+P)/P^2. \tag{5}$$

Since the cross-roll instability forces $q^2 \leq 0.3058$, (Refs. 3 and 4), the zigzag instability is replaced entirely by the former for $P < 10.0$.

The zigzag instability is suppressed for small P since the Ω_z generated by the curvature rotates the rolls back to their rectilinear configuration. In fact the effective equation for φ for $k_x = 0$ is no longer diffusive at long wavelengths but relaxes at a finite frequency. For $q > 0$ and $k_y, k_x \neq 0$, we have recovered the skewed-varicose instability[5] which develops through a slow modulation in both the roll spacing and direction. Our stability diagram near onset for $P < 10.0$ is the mirror image of what was obtained from the old amplitude equation.[3]

The accuracy of (5) as a function of ϵ and P in the ranges $0.1 \leq \epsilon \leq 0.5$ and $10 \leq P \leq 20$, together with the most unstable value of k_y, was checked against a direct numerical simulation of the Boussinesq equations. The theoretical value of q from (5) was generally accurate to 5% except when the cross-roll instability occurred first.

For rigid top and bottom boundary conditions, Ω_z and B_x should still be understood as slowly varying in x-y but now must vary approximately as $z(1-z)$ in the vertical. It is thus natural to replace (1b) by

$$\Omega_z = g' \partial_y[A^*(\partial_x - i\partial_y^2)A + \text{c.c.}], \tag{6}$$

where $\Omega_z = -\partial_y B_x$ and g' is a constant of order $\epsilon^{1/2}$ which varies as P^{-2} for $0.1 \lesssim P \lesssim 1$ and saturates for smaller P. Equation (1a) may be kept invariant by redefining the scale factors. It is the finite dissipation in the Navier-Stokes equation for $\tilde{\Omega}_z$ that forces $\tilde{B}_x \sim O(\epsilon^{3/2})$ and thus makes the last term in (1a) higher order in ϵ than the others, e.g., $B_x \sim O(\epsilon^{1/2})$.[11,12] The condition for the zigzag instability therefore becomes $q < -g'(1-q^2)$, while in a large container the skewed-varicose instability occurs whenever $2qg' > (1-3q^2)/(1-q^2)$. Equation (6) merely corrects the most obvious nonphysical features of the free-slip approximation. It is not systematic in any small parameter but for small P may capture the essentials of the problem; e.g., compare our stability diagrams with Ref. 5.

It is the skewed-varicose and not the oscillatory instability that may well be of greatest relevance to the appearance of time-dependent convection near onset in large containers.[6,13] This problem is most easily approached in the limit $\epsilon, P \to 0$, g' finite, where the dynamics are no longer relaxational. It is thus no longer obvious even for the laterally periodic problem that a system rendered unstable by increasing ϵ for $q > 0$ will ever relax back to the stable states that exist for $q < 0$.[14] Numerical experiments with free-slip boundary conditions have shown that for a small box of order $8\pi/q_0$ square, the system does find its way back to a stable state, whereas for a larger box it does not in a reasonable length of time.

The "turbulence" seen in a large container may resemble a glass in that the noise may be transient but with a time scale that diverges exponentially with the system size. In our numerical experiments for $P = 4$ and $\epsilon = 0.5$, the dynamics was so obviously not relaxational that it is difficult to see why the lateral diffusion time should be relevant. Although our vorticity equation suggests that free-slip boundary conditions accentuate the tendency of a disordered roll pattern to wander in phase space, it is not obvious theoretically why the rather nonuniform textures observed by Gollub and Steinman[13] proved to be so stable just below the skew-varicose transition since the correct dynamics are not derivable from any obvious potential.

Rigid lateral boundaries in x significantly constrain the stationary states accessible to the bulk.[11] We now consider the complimentary problem involving only y variation and argue that unless an acceptable solution can be found to the separate one-dimensional problems a full two-dimensional solution is unlikely to exist. To examine the behavior of rolls subject to a curvature induced by the lateral boundaries within the context of (1a), we consider a wedge-shaped container occupying the region defined in polar coordinates, ρ, θ, by $|\theta| \leq \alpha \ll 1$, $\rho_1 \leq \rho \leq \rho_2$, with $(\rho_2 - \rho_1)/\rho_1 \ll 1$. We set $B_x = 0$, ignore variation in x, and impose the boundary condition $|A| = \partial_y |A| = 0$ at $y = \pm\rho \sin\alpha$.[15] Examination of the amplitude equation in magnitude and phase variables makes the existence of stationary smooth solutions with the rolls normal to the boundaries seem unlikely.

With some assumptions it may be shown that the "free energy" associated with (1a) has a global minimum when the phase of A is a constant. The rolls therefore prefer to remain straight throughout the container. Restoring the B_x field and including (1b) does not appear to generate any new stationary solutions but may instead yield interesting dynamics.

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N-Level Coherence Vector and Higher Conservation Laws in Quantum Optics and Quantum Mechanics

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The presence of unitary group generators in the time evolution of N-level quantum systems is shown to suggest a class of new nonlinear constants of motion, and to permit the description of the evolution in terms of the rotations of a real coherence vector.

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The description of magnetic and optical resonance phenomena, especially at the qualitative level, is enormously simplified by the use of the Bloch spin or pseudospin vector. Unfortunately, the central equation on which the vector description is based, namely

$$d\vec{S}/dt = \gamma\vec{B}\times\vec{S}, \tag{1}$$

is valid only for spins or other physical systems[1] whose energy levels are equally spaced.

It has long been thought impossible[2] to obtain a similarly simple vector description of more complex quantum systems, even including a system as simple as a three-level "atom."[3] Particularly in optical resonance physics, the dynamical evolution of three-level systems is of great importance. It is central to discussions of two-photon coherence,[3,4] resonance Raman scattering and double-resonance processes,[5] three-level superradiance,[6] coherent multistep photoionization and photodissociation,[7] and trilevel echoes.[8]

In this Letter we solve the problem of a vector

