Limitations of Two-Dimensional Model Equations for Ion-Acoustic Waves

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It is shown that plane ion-acoustic solitons are unstable in magnetized plasmas if the magnetic field is greater than a critical value which roughly corresponds to the condition that the soliton width equals the ion gyroradius. On the other hand, ion-acoustic solitons are stable if their amplitudes exceed a corresponding magnetic-field–dependent threshold. Limitations of previous models are discussed.

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It is now well known[1] that the weakly nonlinear one-dimensional description of ion-sound waves in plasmas is given by the classic Korteweg–de Vries (KdV) equation. Solitary-wave solutions of the KdV equation are one-dimensionally fully stable. However, the question arises whether this stability is preserved when weak (two-dimensional) bending distortions are allowed. Two equations were proposed for the description of the process of propagation of a two-dimensional perturbation: Kadomtsev and Petviashvili[2] (KP) allowed for a weak transverse coordinate dependence in an unmagnetized plasma. It was shown[2] that this generalization leads to transversely stable plane-soliton solutions. On the other hand, Zakharov and Kuznetsov[3] (ZK) considered the case of very strong external magnetic fields. According to their equation, plane solitons are unstable.[4] However, recent experiments[5] showed that large-amplitude plane solitons are quite stable even if magnetic fields are present. Because of the various limiting processes (small amplitudes, weak transverse dependence, and vanishing or large magnetic fields) involved in previous theories, a quantitative investigation of this puzzling behavior has to start from a unified description in order to answer the question of existence of threshold in amplitude or magnetic field. This is the first motivation of this Letter. The second stems from a simplification inherent in all previous two-dimensional considerations: The small-amplitude limit[2,3] is not appropriate for most practical applications. We also drop that assumption here in order to study the threshold behavior with respect to finite amplitudes. Thus the stationary states under consideration are those first found by Sagdeev[6] in the Mach-number range $1 < M \lesssim 1.6$. It should be further noted that the two-dimensional model equations discussed here have even broader applications. The KP equation occurs, for example, in shallow-water wave problems; the ZK equation has recently[7] been obtained for vortex solitons of the Hasegawa-Mima[8] type.

We describe the Sagdeev solitons by the hydrodynamic set of equations for ion density n and ion velocity $\vec{v}$. Furthermore, the plasma may be highly nonisothermal ($T_e \gg T_i$) and is situated in a uniform magnetic field $\vec{H} = H\hat{z}$. The characteristic time is assumed to be larger than the ion cyclotron period $2\pi/\Omega_i$. For small β ($\beta = 4\pi n T_e / H^2 \ll 1$) the ambipolar field is a potential field. Then

$$\partial_t n + \nabla \cdot (n\vec{v}) = 0, \tag{1}$$

$$\partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} + \nabla\Phi + \Omega \hat{z} \times \vec{v} = 0, \tag{2}$$

$$\nabla^2 \Phi = \exp(\Phi) - n. \tag{3}$$

Here, we use the following units: Debye length, $\lambda_e = (T_e/4\pi n e^2)^{1/2}$; ion plasma frequency, $\omega_{pi} = (4\pi n e^2/m_i)^{1/2}$; sound velocity, $c_s = (T_e/m_i)^{1/2}$; and potential, T_e/e. The parameter $\Omega = \Omega_i/\omega_{pi}$ can vary from small values to order 1 for practical applications.

One-dimensional stationary solutions of Eqs. (1)–(3) are the Sagdeev solitons[9] (index s); the density n_s and the velocity v_s are functions of the space coordinate z and follow from the potential Φ_s through

$$n_s = (1 - 2\Phi_s/M^2)^{-1/2}, \tag{4}$$

$$v_s = M/n_s, \tag{5}$$

where M is the Mach number. The potential Φ_s follows from the first-order differential equation

$$\tfrac{1}{2}(\Phi_s')^2 = \exp(\Phi_s) + M^2(1 - 2\Phi_s/M^2)^{1/2} - 1 - M^2, \tag{6}$$

where the prime denotes a derivative with respect to z.

We now study the stability of the solutions (4)–(6) with respect to transverse perturbations. For that purpose we use (1)–(3) as basic equations and not any of the weakly nonlinear model equations.[2,3] We apply the powerful Zakharov-Rubenchik procedure[10]; it has the advantage of covering the whole Mach-number range. Expanding all dependent variables in terms of the transverse wave number k, we write for the perturbations

$$\delta n = n_s' + kn_1 + k^2 n_2 + \cdots, \quad \delta\Phi = \Phi_s' + k\Phi_1 + k^2\Phi_2 + \cdots,$$

$$\delta\vec{v} = \begin{pmatrix} 0 \\ 0 \\ v_s' \end{pmatrix} + k\vec{v}_1 + k^2\vec{v}_2 + \cdots .$$

A similar expansion is used for the growth rate, $\gamma = k\gamma_1 + \cdots$.

The first-order equations can be solved explicitly. After some algebra, we find

$$\Phi_1 = \gamma_1 \partial_M \Phi_s, \tag{7}$$

$$v_{z1} = -[\gamma_1(v_s - M) + \Phi_1]/v_s, \tag{8}$$

$$n_1 = -[\gamma_1(n_s - 1) + n_s v_{z1}]/v_s, \tag{9}$$

and

$$v_{x1} = (i/\Omega)\{\sin\psi \int_{-\infty}^{z} d\xi\, [(k_x/k)(v_s v_s')' + \Omega(k_y/k)v_s'] \cos\psi - [\cos\psi \leftrightarrow \sin\psi]\}. \tag{10}$$

The abbreviation $[a \leftrightarrow b]$ means that the first term should be repeated with a and b interchanged. Furthermore, $-v_{y1}$ is obtained from (10) via $[x \leftrightarrow y]$ and $[\Omega \leftrightarrow -\Omega]$. The argument ψ is given in terms of n_s by

$$\psi' = \Omega n_s / M. \tag{11}$$

The growth rate γ_1 follows from the resolvability condition in next order. Using (7)–(10) we find after some algebra

$$\gamma^2 = \frac{\int_{-\infty}^{\infty} [(\Phi_s')^2 - (v_s - M)^2 n_s] dz - \Omega J/M}{\partial_M \int_{-\infty}^{+\infty} M^{-1} n_s (v_s - M)^2 dz} k^2, \tag{12}$$

where

$$J \equiv \int_{-\infty}^{+\infty} dz \int_{-\infty}^{z} d\xi [v_s(z) - M] n_s(z) [v_s(\xi) - M] n_s(\xi) \sin[\psi(\xi) - \psi(z)].$$

Equation (12) is the main mathematical result of this paper. In the following, we shall discuss the physical implications.

First, for vanishing magnetic fields ($\Omega = 0$) the situation is stable. For demonstration, we evaluate (12) in the small-amplitude limit where

$$\Phi_s \simeq 6\eta^2 \operatorname{sech}^2 \eta z \tag{13}$$

and

$$\eta \equiv \tfrac{1}{2}(1 - M^{-2})^{1/2}. \tag{14}$$

Then, immediately,

$$\gamma^2 \simeq -\tfrac{4}{3}\eta^2 k^2 \tag{15}$$

follows, in agreement with the Kadomtsev-Petviashvili result,[2] and no instability occurs.

However, for finite Ω the conclusion drastically changes. To show this, we analytically evaluate J in the limit $\Omega/M\eta \to \infty$. Integrating by parts and using $\cos x = 1 - 2\sin^2(x/2)$ as well as

$$\frac{\sin^2(\Omega\varphi/2M\eta)}{\Omega\varphi^2/2M\eta} \to \pi\delta(\varphi), \tag{16}$$

one obtains after some algebra

$$(\Omega/M)J \to -\int_{-\infty}^{+\infty} (v_s - M)^2 n_s\, dz. \tag{17}$$

Thus, in the numerator on the right-hand side of (12), the third term cancels the second one and, compared with the result for $\Omega = 0$, the numerator of (12) changes sign. That means that for finite Ω and small (but finite) parameters η instability occurs.

In the small-amplitude limit [using Eq. (14)] and for $\Omega/M\eta \to \infty$ we now obtain from (12) the instability growth rate,

$$\gamma^2 \simeq \tfrac{16}{15}\eta^4 k^2. \tag{18}$$

This growth rate agrees with that[4] for the ZK equations[3] for $\Omega/\eta \to \infty$.

Comparing (15) and (18), we conclude that the mathematically correct result of Kadomtsev and Petviashvili[2] is physically only marginally applicable. For any Ω instability occurs if η is small enough! Of course, the quantitative transition from (15) to (18) can only be followed numerically. The details will be published elsewhere[11]; here we present only the numerical results.

The calculations show a continuous transition from stable to unstable behavior. Thresholds in magnetic field strength and amplitude, respectively, occur. In Fig. 1, we depict the instability growth rate γ/k vs Ω for various parameters η. The reverse, γ/k vs η for different Ω, is shown

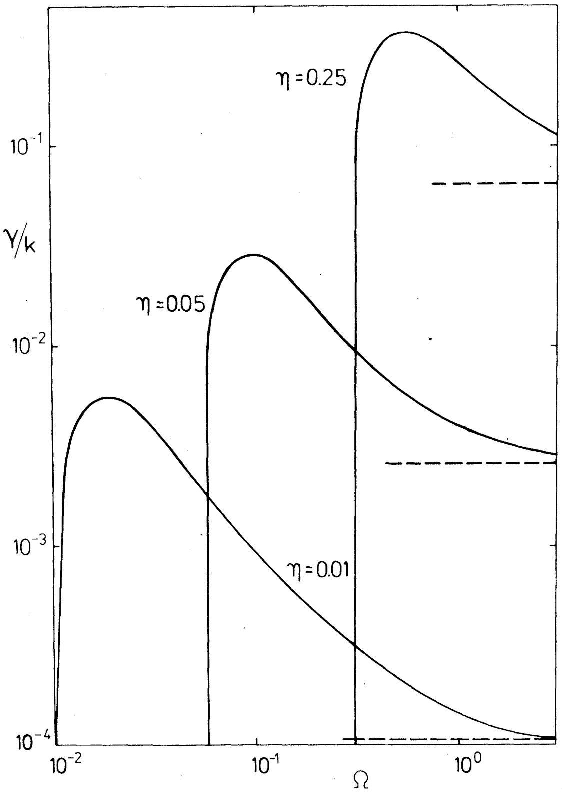

FIG. 1. Growth rate γ/k vs magnetic field strength Ω for three parameters η. The dashed lines depict the corresponding asymptotes for $\Omega/\eta \to \infty$ in agreement with the results of Ref. 4.

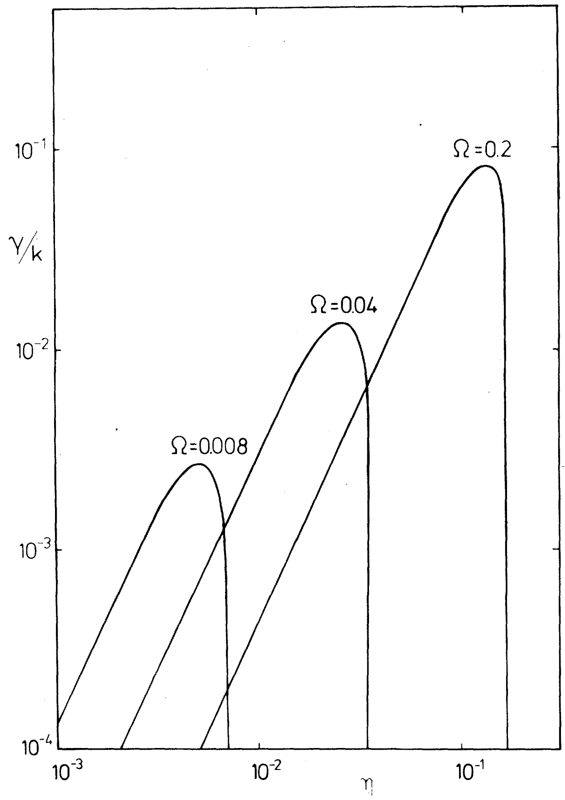

FIG. 2. Growth rate γ/k vs amplitude parameter η for three parameters Ω.

in Fig. 2. The graphs show that growth rates in magnetized plasmas differ from the ZK result ($\Omega/\eta \to \infty$) by an order of magnitude. The calculation allows us to discriminate between stable and unstable regions depending on η and Ω as presented in Fig. 3. Roughly speaking, solitons are stable (unstable) when the width $\lambda_e \eta^{-1}$ is smaller (larger) than the ion gyroradius $\rho_s = c_s/\Omega_i$.

These results can be understood physically as follows. As can be seen from the *linear* part of the dispersion relation, an external magnetic field changes the transverse dispersion properties, i.e., $v_{g\perp} > 0$ (< 0) and $dv_{g\perp}/dk_\perp > 0$ (< 0) for $\Omega = 0$ ($\Omega \to \infty$). Then, in a frame moving with the linear group velocity v_g, any modulation of amplitude A in the transverse direction will decrease (grow) since because of *nonlinear* effects $\partial k_\perp/\partial t \sim -\partial A/\partial x_\perp$ for ω (nonlinear) $\sim A$. Thus locally a transfer of energy occurs to the troughs (crests) of the modulation. The (one-dimensionally stable) solitons will feel the transversely destabilizing effects if the characteristic time for longitudinal

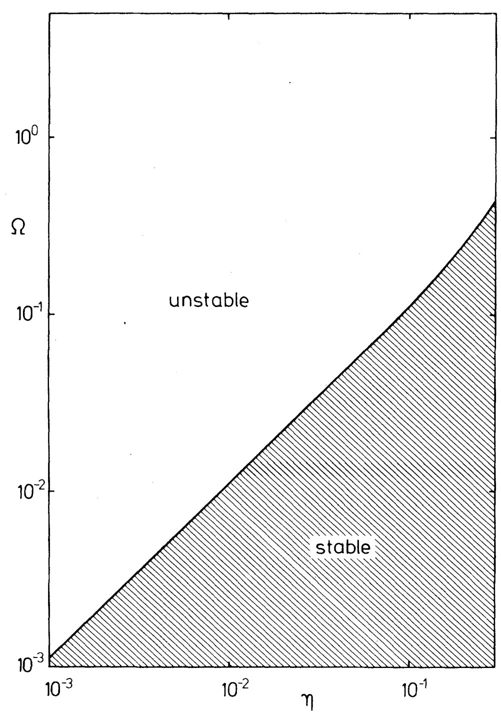

FIG. 3. Borderline of stable and unstable behavior in the (Ω, η) plane.

pressure balance ($\lambda_e \eta^{-1} c_s^{-1}$) exceeds the gyrotime ($\Omega_i^{-1}$), i.e., for $\eta < \Omega$.

In summary, we have shown that the two-dimensional dynamical behavior of ion-acoustic solitons is in general more complicated than assumed previously.[2,3] We have demonstrated that, even for small amplitudes, only in the cases $\Omega = 0$ and $\Omega/\eta \to \infty$ do the KP and ZK model equations apply. For (small but) finite Ω, both limits are not adequate, in general. By detailed numerical computation we have found the transition from stable to unstable behavior. For fixed η, there exists a threshold $\Omega_T \simeq \eta$ in Ω; for $\Omega > \Omega_T$ instability occurs. The instability growth rates are typically larger than those obtained from the ZK equation by a factor 10. On the other hand, for fixed $\Omega \neq 0$, there exists a threshold $\eta_T \simeq \Omega$ in η; for $\eta > \eta_T$ solitons are stable. This explains experimental observability[5] of solitons in magnetized plasmas.

Finally, there are some straightforward (but algebraically complicated) and interesting extensions of the present work. For example, higher-dimensional soliton solutions[12] should be investigated. Future work will be devoted to these topics.

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[9]We note that in a magnetized plasma other types of solitary waves also exist. The stability of the localized solutions propagating obliquely to the magnetic field will be discussed by the same method in a forthcoming paper.

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[11]We should mention that one integration in J can then be performed analytically; however, the algebra becomes quite lengthy. One can show explicitly that the result of instability in the limit $\Omega/M\eta \to \infty$ is correct in any order of η^2. Note that $\eta^2 \lesssim 0.15$ holds because of $1 < M \lesssim 1.6$ and Eq. (14).

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