Quantitive Theory of the Fermi-Pasta-Ulam Recurrence in the Nonlinear Schrödinger Equation

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By limiting attention to the lowest-order Fourier modes we obtain a theory of the Fermi-Pasta-Ulam recurrence that gives excellent agreement with recent numerical results. Both the predicted period of the recurrence and the temporal development of the $n = 0$ mode are very good fits. The maximum of the $n = 1$ mode, however, is off by about 30%. (The nonlinear Schrödinger equation governs the development of the envelope of the electric field of a nonlinear Langmuir wave in the plasma-physics context. It also describes gravity waves in deep water.)

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Recently Yuen and Ferguson[1] demonstrated the Fermi-Pasta-Ulam (FPU) recurrence[2] for the nonlinear Schrödinger equation, which equation is of primary importance in plasma physics and fluid dynamics. The essence of the recurrence phenomenon is that linearly unstable modes demonstrate a "superperiodicity" on a sufficiently long time scale when treated nonlinearly. When this superperiodicity is common to all Fourier modes, the initial conditions will be reproduced every once in a while.

To start the FPU recurrence off we need a linearly unstable mode. In the case of the nonlinear Schrödinger equation this indicates wave number k in a range $0<k<k_c$. Recently Rowlands and Janssen[3] independently used the fact that for k very near k_c calculations can be somewhat simplified. They obtained the long-time behavior of the modulation of the linearly unstable mode in this limit. However, as both authors pointed out, their approach could only lead to a qualitative confirmation of the numerical results of Yuen and Ferguson, as none of the k values used by those authors was sufficiently near k_c. In this Letter we give a more general calculation in which k is not restricted. The results of Rowlands and Janssen are recovered as a limiting case, and the numerical findings of Yuen and Ferguson are described quantitatively as well as qualitatively.

We take the cubic nonlinear Schrödinger equation in the form

$$i(\partial A/\partial t) - \tfrac{1}{8}(\partial^2 A/\partial x^2) - \tfrac{1}{2}|A|^2 A + \tfrac{1}{2}a_0^2 A = 0 \qquad (1)$$

[obtained from the more common form, in which the last term is absent, via the substitution $a = \exp(-ia_0^2 t/2)A$], and assume

$$A = A_0(t) + A_{-1}(t)\exp(-ikx) + A_1(t)\exp(ikx), \qquad A_0(0) = a_0, \quad A_{-1}(0) = A_1(0) = a_1. \qquad (2)$$

For $a_1 = 0$ the constant solution is exact. Our calculation is easily extendable to $A_{-1}(0) \neq A_1(0)$. We neglect the generation of modes $\sim\exp(inkx)$, $n \geq 2$ but do not assume either $k_c - k$ or $|A_1/A_0|$ small. This method is sometimes used in fluid dynamics.[4] (If we did take $|A_1/A_0|$ small and linearized around a_0, we would obtain linear instability of the $n = 1$ modes with $k_c = 2\sqrt{2}a_0$.)

Equations (1) and (2) yield, for the time development of the $n = 0$ mode,

$$i\dot{A}_c + \tfrac{1}{2}a_0^2 A_0 = \tfrac{1}{2}A_0^2 A_0^* + A_{-1}A_1 A_0^* + A_{-1}A_{-1}^* A_0 + A_0 A_1 A_1^* \qquad (3)$$

and two similar equations for the development of A_{-1} and A_1. All three can be obtained from a Lagrangian

$$L = \frac{i}{2}\sum_{l=-1}^{1}(A_l^*\dot{A}_l - \dot{A}_l^* A_l) + \sum_l(\tfrac{1}{2}a_0^2 + \tfrac{1}{8}k^2 l^2)|A_l|^2 - \tfrac{1}{4}\sum_l |A_l|^4 - \sum_{i<j}|A_i|^2|A_j|^2 - \tfrac{1}{2}F, \qquad (4)$$

$$F = A_0^{*2}A_{-1}A_1 + A_{-1}^* A_1^* A_0^2, \quad F(0) = 2a_0^2 a_1^2.$$

Time invariance yields Noether's theorem in the form

$$\sum_{l=-1}^{1}(\dot{A}_l L_{\dot{A}_l} + \dot{A}_l^* L_{\dot{A}_l^*} - L) = \text{const}, \qquad (5)$$

whereas invariance under the phase transformation $A_1 \to A_1 \exp(i\epsilon_l)$, $2\epsilon_0 - \epsilon_1 - \epsilon_1 = 0$ yields the Noether equation

$$\sum_{l=-1}^{1}(\epsilon_l A_l L_{\dot{A}_l} - \epsilon_l A_l^* L_{\dot{A}_l^*}) = \text{const.} \tag{6}$$

Using Eqs. (5) and (6) (the latter in two versions, one with all ϵ_l equal, and the other with $\epsilon_0 = 0$, $\epsilon_1 = -\epsilon_{-1}$) we obtain three distinct conservation laws. These are best expressed in terms of the deviation of $|A_0|^2$ from its initial value,

$$x = a_0^2 - |A_0|^2.$$

They are

$$|A_{-1}|^2 = |A_1|^2 = a_1^2 + \tfrac{1}{2}x,$$

$$F = \tfrac{3}{4}x^2 + (\tfrac{1}{4}r^2 - a_0^2 + a_1^2)x + 2a_0^2 a_1^2.$$

If we now multiply (3) by A_0^* and subtract its complex conjugate, square the result, and express all quantities in terms of x and known constants, we obtain

$$\dot{x}^2 = 4|A_0|^4|A_1|^4 - F^2$$

$$= \tfrac{7}{16}(x - x_1)(x - x_2)(x - x_3)(x - x_4), \tag{7}$$

$$x_1 = (k^2 - 4a_1^2 - \sqrt{\Delta})/2, \quad \Delta = (k^2 - 4a_1^2)^2 + 64a_0^2 a_1^2,$$

$$x_2 = 0, \quad x_2 \le x \le x_3 \text{ when } x_3 > 0,$$

$$x_3 = \tfrac{4}{7}(2a_0^2 - 3a_1^2 - \tfrac{1}{4}k^2),$$

$$\dot{x}_4 = (k^2 - 4a_1^2 + \sqrt{\Delta})/2.$$

Equation (7) can be solved to give

$$x = x_1 x_3 \,\text{sn}^2(m|\alpha t)/[x_3 \,\text{sn}^2(m|\alpha t) + x_1 - x_3],$$

$$m = [x_3(x_4 - x_1)/x_4(x_3 - x_1)]^{1/2},$$

$$\alpha = [7x_4(x_3 - x_1)/8^2]^{1/2}.$$

The superperiod is $T = 2K(m)/\alpha$, and the maximum deviation of $|A_0|^2$ is x_3. For $k^2 > k_c^2 + 2a_1^2$ the FPU recurrence will not occur. This generalizes the cutoff of linear theory k_c^2. There are complications for $k < \frac{1}{2}k_c$ found numerically[1] but not recovered by our theory, as the interplay with higher modes is involved. Finally we note that Janssen's equation [Eq. (16b) of Ref. 3] is recovered from our Eq. (7) by taking $k_c - k \to 0$.

We are now in a position to compare our results with those of Yuen and Ferguson[1] (case 1, for which those authors give the most detail). To do this take $a_0 = 1$, $a_1 = 0.05$, and $k = 2$. Figure 1 gives the comparison. Of course the $n = 2$ and $n = 3$ modes are absent from our theory and that is its main shortcoming. On the other hand the period T is seen to be an excellent fit (within 1%). Neglect of higher-order modes does not seem to affect T at all, no doubt because of enslavement (following of the $n = 1$ mode by those of higher order). Also, and perhaps more surprisingly, the dip in the energy content of the $n = 0$ mode x_3 is also a good fit (correct within a few percent). The maximal energy content in the $n = 1$ modes, on the other hand, is down by about 30%. This is not surprising, as the interplay with higher modes has been neglected.

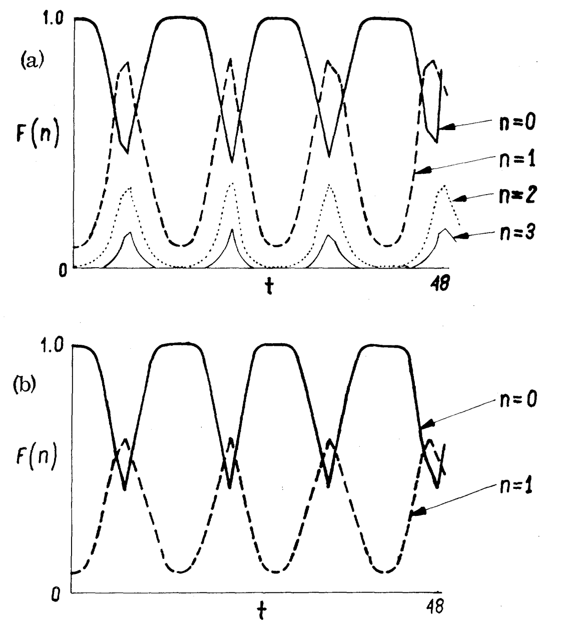

FIG. 1. Normalized energy content of the lowest-order Fourier modes: (a) according to Yuen and Ferguson and (b) as predicted by theory for the same initial conditons. Difference in T cannot be seen on this scale.

In conclusion, we now have a quantitative theory of the Fermi-Pasta-Ulam recurrence for the nonlinear Schrödinger equation, at least for $\frac{1}{2}k_c < k < k_c$. Work on inclusion of $n = 2$, and hence a more realistic treatment of $0 < k < \frac{1}{2}k_c$, is now underway.

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