the presence of the attractive potential. When many of these wavelengths are contained in the interproton distance R, we may expect that the electron wave function around each proton is not much influenced by the other proton and close to the nuclei it may be approximated by the atomic orbitals centered on them. The correlation term in (10) is then equal to 1, and (10) satisfies the obvious condition that the cross section for a projectile composed of two uncorrelated protons is twice that for a single proton.

When the internuclear distance R is not large compared with the wavelength, $\bar{\varphi}_{\vec{\kappa}_1}(R/2, R)$ presents molecular characteristics and differs from $f_C(Z/\kappa_1)$. The local wavelength close to the protons is of order π/Z, and so for $R \ll \pi/Z$ the electron sees a unified charge of value $2Z$ and $\bar{\varphi}_{\vec{\kappa}_1}(R/2, R) \simeq f_C(2Z/\kappa_1)$. Then the correlation factor, close to the peak of the electron distribution, which means $\kappa_1 \ll Z$, is

$$|\bar{\varphi}_{\vec{\kappa}_1}(R/2, R)/f_C(Z/\kappa_1)|^2 \cong 2,$$

when $R \ll \pi/Z$.

We conclude that the ratio between the electron distributions carried by $2H^+$ and H^+ is bound between the values 1 and 2; furthermore, the larger the interproton distance R, the smaller the enhancement of the distribution around $2H^+$ compared with that of H^+. These two features are verified by our measurements.

If we analyze the correlation factor for a fixed R we see that for increasing κ_1 the electronic wavelength decreases and we reach, away from the peak, the region where the electron behaves as in an atomic state around each proton; the correlation factor there is equal to 1. This region is reached for smaller κ_1 when R is increased, and again the experimental results of Fig. 1 verify this feature of the theory.

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Stabilizing Effect of Finite-Gyroradius Beam Particles on the Tilting Mode of Spheromaks

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The equilibrium shape of a low-pressure spheromak plasma with a small component of toroidal current carried by finite-gyroradius particles is computed. The stabilizing influence of this current on the tilting mode is determined by employing an energy principle that includes gyroscopic and finite-gyroradius effects.

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The favorable characteristics of nearly force-free, spherical magnetic configurations[1] dubbed "spheromaks" have led to an enthusiastic vision of a fusion reactor[2] with engineering advantages superior to that of tokamaks. However, Rosenbluth and Bussac[3] have shown that these configurations are subject to the "tilting mode" in which the entire plasma tilts unstably about an axis through the center. The remedy[3] for this mode appears to lie in modifying the plasma shape to one which is oblate and, more importantly, in employing a conducting shell very close to the plasma. Unfortunately, this last requirement detracts considerably from the reactor scenario.

In this paper we show that the inclusion of a component of azimuthal current carried by energetic ions with large gyroradius in the spheromak system leads to (1) an oblateness in the shape of the plasma so that by adjusting this current one could optimize the shape for magnetohydrodynamic stability, and (2) an additional stabilizing effect due to the gyroscopic motion and large angular momentum of these particles. This stabilizing influence can be employed to increase the distance between the plasma and the conducting shroud to improve the reactor prospects.

The analytic investigation is a perturbation expansion about the spheromak configuration by including a small energetic particle current j_b. Thus

$$\nabla\times\vec{B}=k\vec{B}+4\pi j_b\hat{\varphi}, \tag{1}$$

with $j_b=q\int d^3v\, v_\varphi f(H+\Omega P_\varphi)$; H, the particle energy, and P_φ, the canonical angular momentum, are constants of motion, Ω is a positive constant,

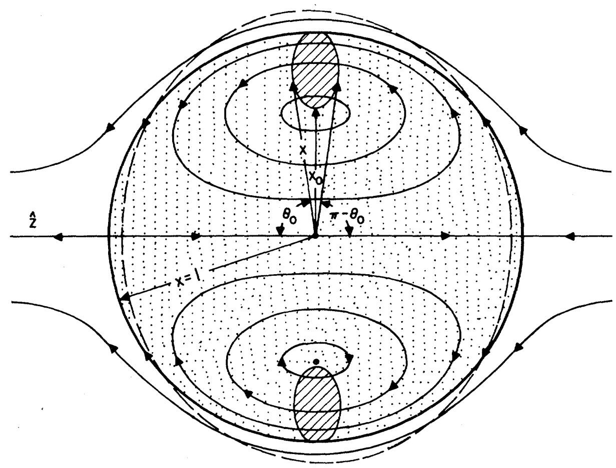

FIG. 1. Magnetic configuration of a spheromak; the cross-hatched region is occupied by the energetic particle current, $\Omega/\Omega_*\sim 1.2$ $(x=r/R)$. Dashed line shows perturbed surface $R+\delta r(\theta)$.

q is the particle charge, and the system is considered to be axisymmetric. The rigid-rotor distribution function is chosen because it maximizes the entropy leading to favorable stability properties. Furthermore, $|I_b/RB_\varphi|\equiv\epsilon\ll 1$ is regarded as an expansion parameter, where I_b is the total particle current. The first-order (in ϵ) perturbed fields are obtained from (spherical coordinates r, θ, φ)

$$\nabla\times\vec{B}^{(1)}=k\vec{B}^{(1)}+4\pi j_b\hat{\varphi}=k\vec{B}^{(1)}-\hat{\varphi}4\pi q\Omega m^{-2}2\pi\int_{-\infty}^{\infty}dP_\varphi\int_V^{\infty}dHf_0(H+\Omega P_\varphi), \tag{2}$$

where $V=(P_\varphi-q\psi_0)^2/2mr^2\sin^2\theta$. The zero-order fields $\vec{B}^{(0)}$ and poloidal flux ψ_0 are those for the spheromak equilibrium.[3] We choose, for calculational convenience, $f=A$ (constant) for $0<H+\Omega P_\varphi<\frac{1}{2}mR^2\Omega^2$ and zero everywhere else. This choice ensures that j_b vanishes at $r=R$, and with $B^{(0)}\to B_*\hat{z}$ for $r\gg R$,

$$j_b=-A(4\pi q/3)R^4\Omega^4x\,|\sin^2\theta|[x^2\sin^2\theta(1+2Y)-1]^{3/2} \tag{3a}$$

where $x=r/R$, $Y=1.54(\Omega_*/\Omega)j_1(kRx)/x$, $j_m(kRx)$ denotes the spherical Bessel function of order m, and $\Omega_*=qB_*/m$. In evaluating j_b which is a quantity of order ϵ, we have used zero-order poloidal flux ψ_0 in f_0. The current density is nonvanishing only between $\theta_0(x)$ and $\pi-\theta_0(x)$ and between x_0 and 1, where $x^2\sin^2\theta_0=(1+2Y)^{-1}$ and $x_0{}^2=(1+2Y)^{-1}$ (see Fig. 1). The constant A can be expressed in terms of the magnitude of the total particle current

$$I_b=\int d^2r\,|j_b|=A(4\pi q/3)R^6\Omega^4\int_{x_0}^1 dx\,x^2\int_{\theta_0}^{\pi-\theta_0}d\theta\sin\theta[x^2\sin^2\theta(1+2Y)-1]^{3/2},$$

$$\equiv A(4\pi q/3)R^6\Omega^4M(\Omega/\Omega_*). \tag{3b}$$

The solution of (2) is obtained from

$$\nabla^2\Phi+k^2\Phi=\begin{cases}-\int^\theta d\theta\,4\pi j_b & \text{for } 0<x<1,\\ 0 \text{ for } x>1,\end{cases} \tag{4}$$

with $\vec{B}^{(1)}=k\vec{r}\times\nabla\Phi+\nabla\times\vec{r}\times\nabla\Phi$. Furthermore, it can be shown for axisymmetric systems that the perturbed poloidal flux

$$\psi^{(1)} = r\sin\theta\, \partial\Phi/\partial\theta. \tag{5}$$

The perturbed shape $r_s = R + \delta r(\theta)$ of the plasma which coincides with the separatrix is obtained from

$$0 = \psi^{(0)}(r_s) + \psi^{(1)}(r_s) \cdots = \psi^{(0)}(R) + \delta r\, \partial\psi^{(0)}/\partial r|_{r=R} + \psi^{(1)}(R). \tag{6}$$

Thus, from (5) and the solution of (4) we obtain for the perturbation of the surface ($\mu \equiv \cos\theta$)

$$\delta r(\theta) = (\tfrac{2}{3}B_*) \sum_{m=1,3,5,\ldots} G_m(R) P_m'(\mu), \tag{7}$$

where B_* is the solenoidal field at infinity,

$$G_m(R) = -(4\pi I_b/M)(m+\tfrac{1}{2}) \int_0^1 dx\, x^3 j_m(kRx) C_m(x)/kRj_{m-1}(kR),$$

and

$$C_m(x) \equiv \int_{-1}^1 d\mu\, P_m(\mu) \int^\theta d\theta\, |\sin\theta| [x^2\sin^2\theta(1+2Y)-1]^{3/2},$$

P_m are the Legendre polynomials, and the prime on P_m denotes a derivative. Note that δr is independent of the sign of k. The $m=1$ term is positive and independent of θ showing a uniform expansion; the $m=3$ term is proportional to $-(15\cos^2\theta - 3)$ and leads to an oblateness of the spheromak. The terms alternate in sign but get smaller and smaller.

To establish the stability of this configuration we employ the energy principle of Sudan and Rosenbluth[4] in the form given by Finn and Sudan[5] and Finn,[6]

$$-\omega^2 T + \omega L + \delta W_p + \delta W_1 + \delta W_2 = 0, \tag{8}$$

where

$$L = \tfrac{1}{2} iq \int d^3r\, n_b \vec{B}\cdot\vec{\xi}^*\times\vec{\xi}, \quad T = \int d^3r\, n_p m_p |\vec{\xi}|^2,$$

$$\delta W_1 = \tfrac{1}{2} q^2\Omega^2 \int d^3r\, d^3v (\partial f_0/\partial H)\rho^2 |\hat{\varphi}\cdot\vec{\xi}\times\vec{B}|^2,$$

$$\delta W_2 = \tfrac{1}{2} i q^2(\omega - l\Omega) \int d^3r\, d^3v\, (\partial f_0/\partial H) g\, dg^*/dt,$$

$$\delta W_p = \tfrac{1}{2}\int d^3r\, (|Q|^2/4\pi - \vec{\xi}^*\cdot\vec{j}_p\times\vec{Q} + \vec{\xi}\cdot\vec{j}_p\times\vec{B}\nabla\cdot\vec{\xi}^* + \gamma p|\nabla\cdot\vec{\xi}|^2),$$

$g = \int_{-\infty}^t dt'\, \vec{\xi}\cdot\vec{v}\times\vec{B}$ is an integral over the orbits of the energetic particles, ω is the eigenfrequency, $\vec{\xi}$ is the plasma displacement $\vec{Q} = \nabla\times\vec{\xi}\times\vec{B}$; in δW_p, $\vec{j}_p$, n_p, and p refer to the plasma current, density, and pressure, respectively, and in a cylindrical coordinate system (ρ,φ,z), $\rho = r\sin\theta$.

Following an argument employed by Rosenbluth and Bussac,[3] we let $\vec{\xi} = \vec{\xi}_0 + \epsilon\vec{\xi}^{(1)}$, where $\vec{\xi}_0 = \theta\vec{r}\times\hat{x}$, a rigid rotation of the configuration about the x axis, is an eigenfunction of the unperturbed spheromak while $\vec{\xi}^{(1)}$ depends upon the distortion caused by j_b; θ is the amplitude of the tilt. If a conducting wall surrounds the plasma at $r = R + \delta r_w(\theta)$, then it has been shown for a force-free spheromak by Rosenbluth and Bussac[3] and under more general conditions by Hammer[7] that $\vec{\xi}^{(1)}$ does not appear in δW_p to first order in ϵ. Omitting the details, we obtain

$$\delta W_p = -\tfrac{3}{4}B_*^2 R^3\theta^2[(RB_*)^{-1} \sum_{m=3,5,\ldots} G_m(R) + \delta r_w/R]. \tag{9}$$

We now proceed to compute the remaining terms in (8) involving gyroscopic and finite-Larmor-radius effects. It is immediately evident that to first order in ϵ these terms can be evaluated with $\vec{\xi}_0$ and $\vec{B}_0$, instead of $\vec{\xi}$ and $\vec{B}$. With $\xi_0 = \theta(-iz, z, ip)\exp[i(\varphi - \omega t)]$ in cylindrical coordinates (ρ,φ,z), the orbit integral g is evaluated as follows:

$$g = \int_{-\infty}^t dt'\, \vec{\xi}_0\cdot\vec{v}\times\vec{B}_0 = (m/q)\int_{-\infty}^t dt'\, \vec{\xi}_0\cdot d\vec{v}/dt' = (m/q)[\vec{\xi}_0\cdot\vec{v} + i\omega\int_{-\infty}^t dt'\, \vec{\xi}_0\cdot\vec{v}].$$

The $i\omega$ term is of order $\omega/\Omega < v_A/R\Omega \ll 1$ and can, therefore, be neglected; v_A is the typical Alfvén speed. Thus, after some lengthy algebra we obtain

$$\delta W_1 + \delta W_2 = (9.24\pi I_b B_* R^2\theta^2/M)\int_{x_0}^1 dx\, x^3 j_1(kRx)\int_{\theta_0}^{\pi-\theta_0} d\theta\, |\sin\theta|\cos^2\theta[x^2\sin^2\theta(1+2Y)-1]^{1/2} \times [x^2\sin^2\theta(\tfrac{5}{2} - Y) - 1]. \tag{10}$$

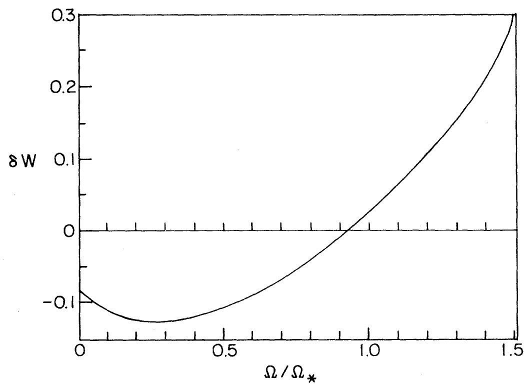

FIG. 2. Plot of $\delta W \equiv (\delta W_1 + \delta W_2)/9.24\pi I_B B_* R^2\theta^2$ as a function of Ω/Ω_*.

The system is stable[8] provided

$$L^2/4T + \delta W_p + \delta W_1 + \delta W_2 > 0. \qquad (11)$$

Although $L^2/4T$ provides a stabilizing influence, in our calculations here it is a quantity of $O(\epsilon^2)$ and will, therefore, be neglected. In Fig. 2, $(\delta W_1 + \delta W_2)/9.24\pi I_b B_* R^2\theta^2$ is plotted as a function of Ω/Ω_*. We keep the uniform field B_*, spheromak radius R, and the total particle current I_b constant. We note that $x_0 \rightarrow 1$ as $\Omega/\Omega_* \rightarrow \frac{3}{2}$, i.e., if the ions have too large an angular velocity they are unconfined. This follows from the fact that ions of energy $\frac{1}{2}mR^2\Omega^2$ are barely confined and their gyrofrequency at $\theta = \pi/2$ is $\frac{3}{2}qB_*/m = \frac{3}{2}\Omega_*$. Thus, for equilibrium to exist $\Omega/\Omega_* \leq \frac{3}{2}$. For $\Omega/\Omega_* < 0.93$, $\delta W_1 + \delta W_2$ becomes negative, i.e., the destabilizing influence of the centrifugal force (δW_1) overcomes the stabilizing effect of the spread in betatron motion (δW_2). Note that $\delta W_1 + \delta W_2$ rises rapidly as $\Omega/\Omega_* \rightarrow \frac{3}{2}$. In the distribution f, the spread in $H + \Omega P_\varphi$ is $-\frac{1}{2}mR^2\Omega^2$, while the mean value is $-\frac{1}{4}mR^2\Omega^2$. It is noteworthy that only the radial field B_r appears in the expression for $\delta W_1 + \delta W_2$ while B_θ determines δW_p. Thus, the tilting mode is independent of the toroidal field B_φ. If δr_w is adjusted to make δW_p vanish, then $\delta W_1 + \delta W_2$ provides a margin of stability. Alternatively δr_w can be increased to take into account the stabilizing effect of $\delta W_1 + \delta W_2$ for $0.93 < \Omega/\Omega_* < \frac{3}{2}$. In passing we mention that the calculation for stabilizing Hill's vortex $(B_\varphi = 0)$ proceeds in a very similar fashion and the expression for $\delta W_1 + \delta W_2$ for this case can be obtained from Eq. (10) by replacing $j_1(kRx)/x$ by $\frac{1}{2}(1 - x^2)$.

The practical situation one looks forward to would probably require the particle current to be of the order of magnitude of the plasma current, i.e., a hybrid particle ring–compact torus. Such equilibria have been obtained numerically[9] and their stability would also have to be tested numerically. It must be noted, however, that for $4\pi I_b \sim B_* R$ the rigid tilt may not be the preferred eigenmode. Numerical simulations of non-field-reversed ion rings[10] show that the ring displacement is mostly in the direction of the external field while it varies in azimuth as $\exp i\varphi$.

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