# Directed Percolation in Two Dimensions: Numerical Analysis and an Exact Solution

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We study directed percolation in two dimensions. Critical properties, such as the threshold $p_c$ and the correlation length exponent $\nu$, depend on the direction $\varphi$ at which percolation is observed. We observe crossover from $\nu(\varphi=0)$ to $\nu(\varphi\neq 0)$; similar crossover is found when the probabilities assigned to vertical and horizontal bonds, $p_V$ and $p_H$, become unequal. For the case $p_H=1$ we solved the problem exactly, obtaining $p_c(\varphi)=[1+\cot(\frac{1}{4}\pi-\varphi)]^{-1}$, $\nu=2$.

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The problem of directed percolation, first posed by Broadbent and Hammersley,[1] is the subject of considerable current interest. Obukhov[2] has shown that directed percolation belongs to a different universality class than nonoriented percolation; the upper critical dimensionality is $d=5$ (rather than $d=6$). Cardy and Sugar[3] established a mapping between directed percolation and Reggeon field theory, relevant to high-energy physics,[4] which, in turn, was shown[5,6] to be related to Markov processes with branching, recombination, and absorption that occur in chemistry and biology.[7] In two dimensions numerical studies which use Monte Carlo,[8] series,[9] and finite-size scaling techniques[10] have recently been performed.

To define a problem of directed percolation,[1] consider a square lattice. To each edge of the lattice a bond either is assigned with probability $p$ or is absent with probability $1-p$. The horizontal bonds carry a left-pointing arrow, and the vertical ones a down-pointing arrow. One wants to calculate the probability $P(\vec{R},p)$ that a pair of sites at $\vec{0}$ and $\vec{R}$ are connected by a path of bonds that can be traversed in the direction of the arrows (see Fig. 1). There exists a value $p_c$ such that for $p<p_c$, $P(\vec{R},p)\sim\exp[-|\vec{R}|/\xi]$ with $\xi\sim(p_c-p)^{-\nu}$. For $p>p_c$ there is one (or more) infinite path. Numerical estimates for $p_c$, $\nu$, and other exponents were obtained for various two-dimensional lattices.[3,8-10]

An important observation[3,6] is the angle dependence of the function $P(\vec{R},p)$. As opposed to ordinary percolation, where $P(\vec{R},p)$ depends on $R=|\vec{R}|$ only, here, for given $R$, the function $P$ varies with the angle $\varphi$ (see Fig. 2). By construction, $P_\varphi(R,p)=0$ for $|\varphi|>45°$. Percolation occurs when, for $p>p_c$, one has $\lim_{R\to\infty}P_\varphi(R,p)=0$ for $|\varphi|>\varphi_c(p)$, and nonzero for $|\varphi|<\varphi_c(p)$. That is, percolation occurs in a "cone" of width $\varphi_c(p)$ centered at $\varphi=0$. The critical angle $\varphi_c(p)$ goes to zero as $p\to p_c^+$.[6] As a direct consequence of this anisotropic behavior of $P$, one can define an angle-dependent correlation length $\xi_\varphi(p)$ for $\varphi>\varphi_c(p)$, $p>p_c$, via $P_\varphi(R,p)\sim\exp[-R/\xi_\varphi(p)]$. This rather unusual anisotropic critical behavior is the subject of our study.

Our findings can be summarized as follows:

(1) The correlation length diverges according to

$$\xi_\varphi(p)\sim[p_c(\varphi)-p]^{-\nu(\varphi)}, \tag{1}$$

i.e., the threshold $p_c(\varphi)>p_c(0)$ and the exponent $\nu(\varphi)$ depends on the angle $\varphi$ at which percolation is observed.

(2) We found crossover from behavior characteristic of $\varphi=0$ to a different type at $\varphi\neq 0$; that

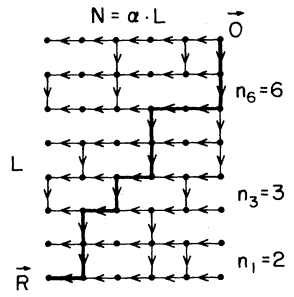


FIG. 1. A particular directed-bond configuration, with all horizontal bonds present ($p_H=1$). The heavy line indicates a proper connecting path from $\vec{0}$ to $\vec{R}$. $n_i$ is the horizontal position of the vertical bond in the $i'$th row that belongs to this path. Other connecting paths are characterized by $n_i'\geq n_i$.

is, for all $\varphi \neq 0$, $\nu(\varphi) = \nu_1 \neq \nu(0)$.

(3) The same behavior is obtained at fixed $\varphi = 0$, when different probabilities $p_H$ and $p_V$ are assigned to horizontal and vertical bonds; $\nu(p_H \neq p_V) = \nu_1 \neq \nu(p_H = p_V)$.

(4) For the case $p_H = 1$, we found for any $|\varphi| < \pi/4$

$$p_c(\varphi) = [1 + \cot(\pi/4 - \varphi)]^{-1}; \quad \nu_1 = 2. \tag{2}$$

Results 1, 2, and 3 were obtained by calculating $P_\varphi(R, p_H, p_V)$ by the transfer-matrix method, for infinite strips of finite width, and with use of finite size scaling. Results 4 are obtained from the exact analytic solution of the $p_H = 1$ problem.

We start with the finite-size scaling method[11] to estimate $P_c(\varphi)$ and $\nu(\varphi)$. This technique has been recently applied to ordinary[12] and directed[10] percolation. The correlation length $\xi_\varphi(p, 1/N)$ is calculated from the largest eigenvalue $\lambda_0$ of the transfer matrix $T_\varphi(S, S')$, $\xi_\varphi^{-1} = \ln\lambda_0$. $T_\varphi(S, S')$ is defined on a strip of width $N$ with fixed angle $\varphi$ as shown in Fig. 2. $T_\varphi(S, S')$ gives the probability that, say, row $M+1$ is in state $S$ if row $M$ is in state $S'$. Here $S$ and $S'$ are just Ising-like states, i.e., a given site $i$ has the value $+1$ if it is connected to row 1 by a path of bonds, or $-1$, if it is not connected. Thus $T_\varphi(S, S')$ is a matrix of size $2^N \times 2^N$; it can be written as a product of $N$ sparse matrices, each having $2 \times 2^N$ nonzero elements only. The largest eigenvalue $\lambda_0$ is found by numerical iteration. We calculate $\lambda_0$ for strips of width $N$ up to $N = 15$.

If $N$ is large and the system is near criticality, $\xi_\varphi$ obeys the scaling relation

$$\xi_\varphi(\delta p, 1/N) = b^\theta \xi_\varphi(b^{\theta/\nu}\delta p, b/N), \tag{3}$$

where $b$ is a change of length scale in the (finite) direction of the rows, $\delta p = (p_c - p)/p_c$ and $\theta$ is the anisotropy exponent.[10] $p_c$, $\theta$, and $\nu$ are calculated by Eq. (3) with use of three strips of width $N$, $N-1$, and $N-2$. For $N \leq 15$, the results still depend on $N$ to some extent, and extrapolation to $N \to \infty$ is obtained by fitting the $N$ dependence by a quadratic form in $1/N$.

Table I shows the results[13] for $\tan\varphi = 0$, $\frac{1}{3}$, and $\frac{1}{2}$. For $\varphi = 0$ the results are in agreement with those of series expansion for directed percolation,[9] and Reggeon quantum spin models,[4] and of Monte Carlo calculations[6] for branching Markov processes.

Our results show that the percolation threshold $p_c$ becomes larger with increasing $\varphi$. This means that one has to increase the number of bonds until the percolating cone goes through the direction of observation. Furthermore, Table I shows that $\varphi = 0$ critical exponents are very different from the $\varphi \neq 0$ ones. However, within the error bars we see the same exponents $\theta$ and $\nu$ for both of the nonzero values of $\varphi$ considered. This indicates a crossover from the $\varphi = 0$ values to universal ones for all $\varphi \neq 0$.

Thus we see a different critical behavior of the correlation length when observed in or off the direction of the percolating cone. In order to check whether this is a general result we have rotated the percolating cone and then calculated the $\varphi = 0$ correlation length. Obviously the cone can be rotated by allowing $p_H \neq p_V$, where $p_H$ and $p_V$ are the probabilities of horizontal and vertical bonds, respectively. From the scaling of $\xi_{\varphi=0,\lambda}(p, 1/N)$, Eq. (3), where $p_H = p$, $p_V = \lambda p$, we obtain the $p$-$\lambda$ phase diagram shown in Fig. 3. The exponent $\nu$ for $\lambda = 1$ corresponds to the $\varphi = 0$ results mentioned above.[10] For $\lambda \neq 1$ we find a crossover to the $\varphi \neq 0$ exponents of Table I; for both $\lambda = \frac{3}{4}$ and and $\frac{1}{2}$ we get $\theta \simeq \nu = 2.0 \pm 0.1$. This supports the idea that the exponents $\nu$ and $\theta$ which describe the divergence of a correlation length outside of the percolating cone are universal.

We now present the exact solution to the prob-

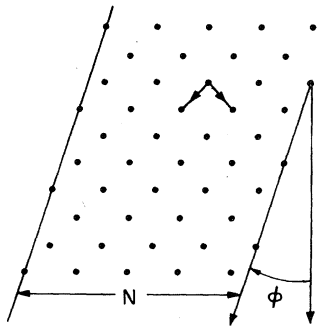


FIG. 2. An infinite strip of width $N$, oriented at $\varphi$ from the diagonal, used in the transfer-matrix method.

TABLE I. Angle-dependent percolation threshold $p_c$, anisotropy exponent $\theta$, and correlation length exponent $\nu$ for directed bond-site percolation with $p_s = p_b = p$ ($p_b$ and $p_s$ are the probabilities of occupied bonds and sites, respectively).

| $\tan\varphi$ | $p_c$ | $\theta$ | $\nu$ |
|---|---|---|---|
| 0 | 0.8228(1) | 1.583(1) | 1.732(3) |
| 1/3 | 0.8582(4) | 2.01(1) | 2.03(2) |
| 1/2 | 0.8871(3) | 2.004(2) | 2.04(10) |

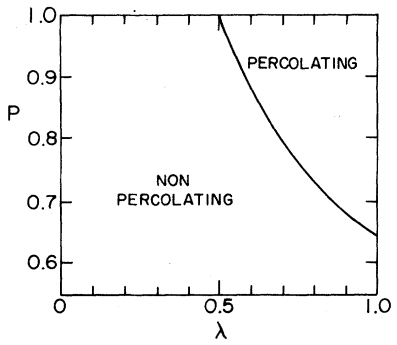


FIG. 3. Phase diagram of a system with horizontal bond probability $p_H = p$ and vertical bond probability $p_V = \lambda p$. Percolation is observed at $\varphi = 0$.

lem with $p_H = 1$, $0 \leq p_V = p \leq 1$. This two-dimensional percolation problem can be expressed in terms of a one-dimensional model, and therefore the solution is quite simple.

Consider an infinite lattice. Choose any site as the origin $\vec{0}$ and consider another site, $\vec{R}$, at $N-1$ steps to the left and $L$ steps down. We calculate the probability $P_{N,L}(p)$ that a properly directed path, such as that of Fig. 1 connects $\vec{0}$ and $\vec{R}$. To avoid traversing the path in a direction opposite to the arrows (that point left or down) a proper path has to stay within the rectangle whose opposite corners are $\vec{0}$ and $\vec{R}$. Such a path can be characterized by a set of $L$ integers $\{n_i\}$. $n_i$ denotes the horizontal location of the first (i.e., closest to the "target" corner $R$) vertical bond present in the first row from bottom. $n_2$ is the location of the first vertical bond present in the second row, such that $n_2 \geq n_1$, and so on; the constraint on the $L$th vertical bond is $n_L \leq N$.

Note that only in our special case of directed percolation, with $p_H = 1$, is it sufficient and possible to parametrize a connecting path in this manner. (If $p_H < 1$, for some bond configurations a connecting path may be drawn only through the second, rather than the first vertical bond present in the first row.) The probability for such a path is given by

$$P_{L,N}(\{n_i\};p) = p^L(1-p)^{n_1-1}(1-p)^{n_2-n_1}\cdots(1-p)^{n_L-n_{L-1}} = p^L(1-p)^{n_L-1} \tag{4}$$

and $P_{L,N}(p)$ is given by summing over all $n_i$ such that $1 \leq n_i \leq n_{i+1} \leq N$, i.e.,

$$P_{L,N}(p) = p^L \sum_{n_L=1}^{N} (1-p)^{n_L-1} \sum_{n_{L-1}=1}^{n_L} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} 1 = p^L \sum_{n_L=1}^{N} (1-p)^{n_L-1} I_{L-1}(n_L). \tag{5}$$

The coefficients $I_L(m)$ are given by[14]

$$I_L(m) = \binom{L+m-1}{L}. \tag{6}$$

To see this, note that the coefficients $I_L(m)$ satisfy the recursion relation $I_L(m) = I_L(m-1) + I_{L-1}(m)$ by their definition (5); the same holds also for $\binom{L+m-1}{L}$. This together with direct evaluation of $I_1(m)$ and $I_2(m-1)$ shows that (6) holds for all $L$ and $m$. In order to calculate $P_{L,N}(p)$ by (5) and (6), denote $x = 1-p$, and note that the function $P_{L,N}(x)$ satisfies the relation

$$dP_{L,N}(x)/dx = -(L+N-1)I_{L-1}(N)x^{N-1}(1-x)^{L-1}. \tag{7}$$

Integrating (7) and using $P_{L,N}(x=0) = 1$, we get by the method of steepest descent, for $L, N \to \infty$,

$$P_{L,N}(x) = 1 - A\int_{-BL^{1/2}}^{B\epsilon L^{1/2}} \exp(-\tfrac{1}{2}\eta^2)d\eta, \tag{8}$$

where $B = [\alpha(1+\alpha)]^{1/2}$, $\alpha = N/L$, $\epsilon = (x-x_c)/x_c$, $x_c = \alpha/(1+\alpha)$, $A = (N+L+1)I_L(N+1)x_c[x_c^{\alpha}(1-x_c)]^L/BL^{1/2}$. When Stirling's formula is now used for the factorials in $I_L(N+1)$, we obtain, in the limit $L, N \to \infty$,

$$\lim_{L\to\infty} P_{L,N=\alpha L}(p) = \begin{cases} 0, & p < p_c(\alpha) \\ \frac{1}{2}, & p = p_c(\alpha) \\ 1, & p > p_c(\alpha), \end{cases} \tag{9}$$

with $p_c(\alpha) = [1+\alpha]^{-1}$; with our definitions of $\alpha$ and $\varphi$ one has $\tan(\frac{1}{4}\pi - \varphi) = 1/\alpha$.

Since $p_H = 1$, even for very small $p_V = p$ the system always percolates at directions sufficiently close to horizontal ($\varphi = 45°$). To obtain $\nu$, note that $P_{L,N}(1-p)$ as given by (8) is already in scaling form; as $L \to \infty$, $A \to (2\pi)^{-1/2}[1 + O(1/L)]$ and the only dependence on $\epsilon \propto p - p_c$ comes through the upper limit of the integral, i.e., $\epsilon L^{1/2}$. Therefore length scales like $\epsilon^{-2}$; explicitly, from $\xi^{-1} = -\lim_{L\to\infty} \ln(P_{L,\alpha L}/L)$ we get $\xi(\alpha, p) \sim [p_c(\alpha) - p]^{-2}$, that is, $\nu = 2$.

From Eqs. (8) and (9) we see the seemingly curious result that for $p > p_c$, $\lim_{L\to\infty} P_{L,\alpha L} = 1$, while $P_{L,\alpha L} < 1$ for finite $L$. This feature results from our choice $p_H = 1$, $p_V < 1$; with such choice, the ordinary (nondirected) percolation problem exhibits

analogous behavior as well. That is, for an infinite system, for $p_V > p_c = 0$ one will have $P_{L,\alpha L} = 1$ for all $L$, whereas for a finite system $P_{L,\alpha L} < 1$. By definition of the directed-percolation problem, keeping $L$ finite corresponds to working with a finite lattice. Also, since for ordinary percolation with $p_H = 1$, $\lim_{L\to\infty} P_{L,\alpha L}$ "jumps", to the value 1 for $p > p_c = 0$, clearly this jump is not related to the standard definition of the exponent $\beta$.

We have calculated $\nu$ exactly for the special point $p_H = 1$ of the phase diagram of Fig. 3. As mentioned before, the numerical analysis indicates that $\nu = 2$ holds for the general case $p_H \neq 1$, $p_H \neq p_V$, too.

We want to mention that Eq. (9) for $\alpha = 1$ can also be obtained from a duality transformation which maps a system with $p_H = 1$, $p_V$ to that on the dual lattice with $\tilde{p}_H = 1 - p_V$ and $\tilde{p}_V = 1$.

Note also that $(1-x)^L p_{L,N}(x)$, Eq. (4), is the partition function of $L$ particles on a chain with $N$ sites, with nearest-neighbor interactions $E_{i,i\pm1} = -k_B T \ln x |n_i - n_{i\pm1}|$. From Eq. (8) one easily obtains that the free energy is zero for $x \leq x_c(\alpha)$ and increases as $\xi^{-1}$ for $x \gtrsim x_c(\alpha)$. Thus this system has a second-order phase transition at $x_c(\alpha)$ with a jump in the specific heat.

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[13] Actually, because of better convergence of $\nu(N)$, the results shown are mixed site-bond percolation with $p_s = p_b = p$. However, since for $\varphi = 0$ $\nu$ is universal along the $p_s$ ($p_b$) phase boundary (see Ref. 10), we expect the same to be true for $\varphi \neq 0$.

[14] We thank Y. Frishman for pointing this out.