Local Scalar Fields Equivalent to Nambu-Goto Strings

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We prove the mathematical equivalence of Nambu-Goto strings to local scalar fields $S(x)$ and $T(x)$ described by the Lagrangian $\mathfrak{L}=-\int d^4x\,\{[\partial(S,T)/\partial(x_\mu,x_\nu)]^2/2\}^{1/2}$. Implications to the quantization problem of strings are also discussed.

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Nambu-Goto string theory[1] was originally proposed as a model of hadrons to give foundation to the successful Veneziano amplitudes for meson-meson scattering at low energies.[2] In the past ten years, quantum chromodynamics (QCD) has emerged as a basic theory of strong interactions, which forces us to explain how QCD is related to the string model. One can argue that a bag picture, which reduces to the string picture in a certain limit, might emerge from QCD by taking account of special configurations of gauge fields.[3] It is also possible to discuss conditions for magnetic superconductivity, expected to be a responsible mechanism for electric string formation, to show that a new kind of permeability in QCD must vanish in the vacuum.[4] In these approaches QCD and the string model are rather indirectly related to each other with only the common feature of the stringlike structure of hadrons. Recently many attempts have been made to connect QCD more directly with the Nambu-Goto strings. For instance, it has been argued that a path-ordered phase factor in QCD obeys an equation of motion quite similar to that in the string model.[5] Another link may be found by showing that the string model is equivalent to a nonlinear Abelian gauge theory of a certain kind, which realizes some of the important properties of QCD.[6-8] Our approach is similar to the last one. We shall prove mathematical equivalence of the string model to a local nonlinear scalar field theory, to which QCD is supposed to eventually reduce at low energies.

Nambu-Goto strings $x_\mu(\tau,\sigma)$ are described by the action

$$I_{\rm string}=-(2\pi\alpha')^{-1}\int d\tau\,d\sigma\,(-v_{\mu\nu}{}^2/2)^{1/2}, \tag{1}$$

$$v_{\mu\nu}=\{x_\mu,x_\nu\}=\partial(x_\mu,x_\nu)/\partial(\tau,\sigma), \tag{2}$$

which gives the equation of motion

$$\{x^\mu,p_{\mu\nu}\}=0, \tag{3}$$

$$p_{\mu\nu}=\frac{\delta I_{\rm string}}{\delta v_{\mu\nu}}=\frac{1}{2\pi\alpha'}\,\frac{v_{\mu\nu}}{(-v^2/2)^{1/2}}. \tag{4}$$

To establish a relation between strings and scalar fields we need to consider a family of solutions[8] to (3) parametrized by (S,T) such that $x_\mu(\tau,\sigma;S,T)$ sweeps a four-dimensional domain Ω in the Minkowski space-time when $(\tau,\sigma;S,T)$ covers a domain Ω'. The induced mapping $(\tau,\sigma;S,T)\to x_\mu$ yields an identity

$$v_{\mu\nu}=\chi^{-1}W_{\mu\nu}=\chi^{-1}(\tfrac{1}{2}\epsilon_{\mu\nu\rho\lambda}W^{\rho\lambda}), \tag{5}$$

$$W_{\mu\nu}=\frac{\partial(S,T)}{\partial(x^\mu,x^\nu)},\quad \chi=\frac{\partial(\tau,\sigma,S,T)}{\partial(x_0,x_1,x_2,x_3)}. \tag{6}$$

Here we have regarded τ, σ, S, and T as functions of a space-time point x in Ω. We now demonstrate that the string theory (1) is equivalent at the classical level to a theory of local scalar fields $S(x)$ and $T(x)$ with the action

$$I_{\rm scalar\ fields}=-\int d^4x\,(W_{\mu\nu}{}^2/2)^{1/2}. \tag{7}$$

To see this, first note that $p_{\mu\nu}$ in (4) may be written with the aid of (5) as

$$p_{\mu\nu}=(2\pi\alpha')^{-1}\tilde{W}_{\mu\nu}/(W^2/2)^{1/2}. \tag{8}$$

Since $p_{\mu\nu}$ may be viewed as a function of x defined in Ω, we can rewrite the equation of motion (3) as

$$\begin{aligned}\{x^\mu,p_{\mu\nu}\}&=\{x^\mu,x^\lambda\}p_{\mu\nu,\lambda}\\&=2\pi\alpha'(-v^2/2)^{1/2}p^{\mu\lambda}p_{\mu\nu,\lambda}=0.\end{aligned}$$

Here $p_{\mu\nu,\lambda}=\partial_\lambda p_{\mu\nu}$. Furthermore, as

$$\begin{aligned}&p^{\mu\lambda}p_{\mu\nu,\lambda}\\&\quad=\tfrac{1}{2}p^{\mu\lambda}(p_{\mu\nu,\lambda}+p_{\nu\lambda,\mu}+p_{\lambda\mu,\nu})-\tfrac{1}{2}p^{\mu\lambda}p_{\lambda\mu,\nu}\\&\quad=-\tilde{p}_{\nu\lambda}\tilde{p}^{\lambda\mu}{}_{,\mu}+\tfrac{1}{4}\partial\nu(p_{\mu\nu}p^{\mu\lambda})\end{aligned}$$

and $p_{\mu\nu}{}^2=$const, the equation of motion is cast into

$$\tilde{p}_{\nu\lambda}\tilde{p}^{\lambda\mu}{}_{,\mu}=0. \tag{9}$$

Substituting (8) into (9), we obtain

$$W_{\nu\lambda}\partial\mu\{W^{\lambda\mu}/(W^2/2)^{1/2}\}=0,$$

which gives

$$\partial_\mu S\,\partial_\nu\{W^{\mu\nu}/(W^2/2)^{1/2}\}=0, \tag{10a}$$

$$\partial_\mu T\,\partial_\nu\{W^{\mu\nu}/(W^2/2)^{1/2}\}=0. \tag{10b}$$

(Here we have assumed that $v^2\neq 0$ and $W^2\neq 0$ except at singularities or boundaries.) Equations (10a) and (10b) are exactly the equations of motion derived from the action (7) by taking variation over S and T. That is, a family of minimal surfaces described by (1) has one-to-one correspondence with a configuration of classical scalar fields described by (7).

Correspondence between strings and local field theories has been previously investigated by several authors. Nielsen and Olesen[9] have stated that Nambu-Goto strings have close resemblance to an Abelian gauge theory with the Lagrangian $-(F^2/4)^{1/2}$ ($F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$), which yields equations of motion

$$\partial_\mu\{F^{\mu\nu}/(F^2/4)^{1/2}\}=0. \tag{11}$$

The system (7) can be partially regarded as Nielsen and Olesen's gauge theory by identifying $A_\mu=S\partial_\mu T$, but, in view of (10a) and (10b), Eq. (11) is too restrictive to describe general motion of strings. On the other hand, Nambu[7,8] and Rinke[6] have tried to find a relation between gauge fields and strings in the form $F_{\mu\nu}=\lambda v_{\mu\nu}$, or $\lambda p_{\mu\nu}$, which sometimes fails to hold as $\tilde{p}^{\lambda\mu}{}_{,\mu}\neq 0$ in general. In other words, the equation of motion requires only Eq. (9), but not $\tilde{p}^{\lambda\mu}{}_{,\mu}=0$.

It might be worth noting that the equation of motion for strings, (9), can be interpreted, on identifying $\tilde{p}_{\mu\nu}=F_{\mu\nu}$, as implying vanishing Lorentz force for electric charges.

It is easy to rewrite the nonlinear scalar field theory (7) in the canonical form. In terms of momenta conjugate to $S(x)$ and $T(x)$

$$\begin{aligned}\Pi_S&=(W^2/2)^{-1/2}\nabla T(\dot S\nabla T-\dot T\nabla S),\\ \Pi_T&=-(W^2/2)^{-1/2}\nabla S(\dot S\nabla T-\dot T\nabla S),\end{aligned} \tag{12}$$

the Hamiltonian is given by

$$H=\int d^3x\,\{(\Pi_S\nabla S+\Pi_T\nabla T)^2+(\nabla S)^2(\nabla T)^2-(\nabla S\nabla T)^2\}^{1/2}, \tag{13}$$

which is positive definite. Canonical equations

$$\begin{aligned}\dot S(x)&=\delta H/\delta\Pi_S(x),\\ \dot\Pi_S(x)&=-\delta H/\delta S(x),\quad\text{etc.},\end{aligned} \tag{14}$$

reproduce (10a) and (10b).

Some examples are in order. First consider spinning strings (rotating bars) described by

$$\begin{aligned}&x_0=\tau,\quad x_1=\cos(\omega\tau+\delta)f(\sigma),\\ &x_2=\sin(\omega\tau+\delta)f(\sigma),\quad x_3=z,\quad \omega=\text{const}\end{aligned} \tag{15}$$

with $f(0)=-\omega^{-1}$ and $f(\pi)=\omega^{-1}$. Clearly the family of the solutions is parametrized by z and δ. By introducing a cylindrical coordinate system (t,r,φ,z) we can immediately write the corresponding solution in the scalar field theory (7):

$$S(x)=z,\quad T(x)=\varphi-\omega t. \tag{16}$$

It is easy to see that (16) indeed satisfies (10).

Another example is a rotating bow with its two ends kept fixed, namely[10]

$$\begin{aligned}&x_0=\tau,\quad x_1=d\cos(\omega\tau+\delta)\sin(\pi\sigma/L),\\ &x_2=d\sin(\omega\tau+\delta)\sin(\pi\sigma/L),\quad x_3=\sigma,\\ &d^2=\omega^{-2}-(L/\pi)^2.\end{aligned} \tag{17}$$

Solutions are parametrized by d^{-1} and δ. The corresponding solutions in the scalar field theory are, in the cylindrical coordinates,

$$\begin{aligned}&S=d^{-1}=r^{-1}\sin(\pi z/L),\quad T=\delta=\varphi-\omega t,\\ &\omega=\frac{\sin(\pi z/L)}{[r^2+(L/\pi)^2\sin^2(\pi z/L)]^{1/2}}.\end{aligned} \tag{18}$$

Rather lengthy calculations confirm that (18) really satisfies (10). Note also that $\partial^\mu\tilde{p}_{\mu\nu}\neq 0$ in this example.

The equivalence of the string theory to the scalar field theory reminds us of the Hamilton-Jacobi formalism for point particles, in which a free point particle is described by a field equation $(\partial S)^2+m^2=0$. In fact, Eguchi[11] has proposed an equation for a string of this sort in which an area swept by the string plays a role of an evolution parameter, in a quite analogous way as the time variable t does in point particle mechanics. On the other hand, Nambu[7] has proposed a Hamilton-Jacobi–type formalism for strings where two evolution parameters τ and σ are treated on an equal footing. In his scheme, string dynamics is described by a first-order partial differential equation for two kinds of scalar fields defined in the six-dimensional manifold (x_μ,τ,σ). In spite of considerable similarity of our Eqs. (10) to the Hamilton-Jacobi equation in point particle mechanics, there is a crucial difference between them, in that Eqs. (10) are second-order partial differential equations.

Our discussions above are entirely classical. One may now deal with the quantization problem of strings by pushing forward the analogy to the Hamilton-Jacobi equations further. It is well known that quantum theory carries dual pictures, namely, the particle and wave pictures. In the point-particle case the Hamilton-Jacobi equation $(\partial S)^2+m^2=0$, which determines trajectories of a particle, is obtained from the Klein-Gordon wave equation $(\hbar^2\partial^2-m^2)=0$ in the $\hbar\to 0$ limit by identifying $\varphi=a\exp(-iS/\hbar)$. In a parallel way one can ask what a wave equation for strings is which yields classical string equations (10) in the $\hbar\to 0$ limit. The answer is quite simple. Consider two kinds of complex scalar fields $\theta(x)$ and $\psi(x)$ described by the Lagrangian

$$\mathcal{L}=\{Z_{\mu\nu}{}^\dagger Z^{\mu\nu}/(\theta^\dagger\theta\psi^\dagger\psi)\}^{1/2},\qquad Z^{\mu\nu}=\partial(\theta,\psi)/\partial(x_\mu,x_\nu). \tag{19}$$

Equations of motion are

$$\partial_\mu\left\{\frac{Z^{\mu\nu}\partial_\nu\psi^\dagger}{\mathcal{L}\theta^\dagger\theta\psi^\dagger\psi}\right\}+\frac{\mathcal{L}}{2\theta^\dagger}=0,\qquad \partial_\mu\left\{\frac{Z^{\mu\nu}\partial_\nu\theta^\dagger}{\mathcal{L}\theta^\dagger\theta\psi^\dagger\psi}\right\}+\frac{\mathcal{L}}{2\psi^\dagger}=0. \tag{20}$$

By making an *Ansatz* that the wave functions θ and ψ in an almost classical physical system have the form

$$\theta=u\exp(iS/\hbar),\quad \psi=v\exp(iT/\hbar), \tag{21}$$

we can ascertain that Eqs. (20) reduce to Eqs. (10) to the leading order in $\hbar$, namely to $O(\hbar^{-1})$. [Note that terms of $O(\hbar^{-2})$ cancel each other out.]

$S(x)$ and $T(x)$ are phases of $\theta(x)$ and $\psi(x)$. This interpretation is quite reasonable in view of the character of multivaluedness of the solutions (16) and (18). By requiring θ and ψ to be single-valued functions of x, we have, for instance, a solution

$$S(x)=n\hbar z,\quad T(x)=n\hbar(\varphi-\omega t)\quad (n=1,2,\ldots). \tag{22}$$

The meaning of the quantization condition in (22) remains to be investigated further.

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