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Oscillatory Phenomena and Q Switching in a Model for a Laser with a Saturable Absorber

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Sufficiently long population decay times and sufficiently short dipole decay times in a single-mode laser with saturable absorber permit passive Q switching in the form of a hard-mode sustained relaxation oscillation.

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A saturable absorber in a cavity permits emission of trains of pulses (passive Q switching).[1,2] We show that this phenomenon appears as a limit-cycle oscillation in a model of a laser with absorber.[3,4] Using the notation introduced in Ref. 3 we have

$$\dot{E} = Nv + \overline{N}\overline{v} - \kappa E, \tag{1a}$$

$$\dot{v} = |g|^2 DE - \gamma_\perp v, \tag{1b}$$

$$\dot{\overline{v}} = |g|^2 \overline{D} E - \overline{\gamma}_\perp \overline{v}, \tag{1c}$$

$$\dot{D} = -\gamma_\parallel D - 4vE + \gamma_\parallel \sigma, \tag{1d}$$

$$\dot{\overline{D}} = -\overline{\gamma}_\parallel \overline{D} - 4\overline{v}E + \overline{\gamma}_\parallel \overline{\sigma}. \tag{1e}$$

A dot over a quantity indicates time derivative. E is the electric field; D is the atomic inversion; v is the polarization; $\gamma_\parallel$ and $\gamma_\perp$ are longitudinal and transverse relaxation rates, respectively; κ is the photon decay rate in the cavity, $\kappa = c(1-R)/L$, where R is the reflectivity of the mirror at the boundaries and L is the length of the cavity; σ is the unsaturated inversion; N is the number of atoms; and $|g|$ is the field-matter coupling constant. A bar over a quantity refers to the passive atoms. c is the velocity of light.

We now write

$$E = -(\gamma_\parallel \gamma_\perp)^{1/2} a / 2|g|,$$

$$v = -\sigma |g| (\gamma_\parallel / 4\gamma_\perp)^{1/2} p, \quad \overline{v} = -\overline{\sigma} |g| (\gamma_\parallel / 4\gamma_\perp)^{1/2} \overline{p},$$

$$D = \sigma(1-d), \quad \overline{D} = \overline{\sigma}(1-\overline{d}), \quad t' = \gamma_\perp t, \quad \omega = \gamma_\parallel/\gamma_\perp,$$

$$r_1 = \overline{\gamma}_\perp/\gamma_\perp, \quad r_2 = \overline{\gamma}_\parallel/\gamma_\parallel, \quad \rho = \kappa/\gamma_\perp,$$

$$A = N|g|^2\sigma/\kappa\gamma_\perp,$$

and

$$C = 1 - \overline{N}|g|^2\overline{\sigma}/\kappa\gamma_\perp.$$

We obtain the following dimensionless equations:

$$\dot{a} = \rho[-a + Ap + r_1(1-C)\overline{p}] \tag{2a}$$

$$\dot{p} = a(1-d) - p \tag{2b}$$

$$\dot{\overline{p}} = a(1-\overline{d}) - \overline{p} r_1 \tag{2c}$$

$$\dot{d} = \omega(-d + ap) \tag{2d}$$

$$\dot{\overline{d}} = \omega(-r_2 \overline{d} + a\overline{p}). \tag{2e}$$

We set $r_2 = 1$, i.e., we take the longitudinal relaxation time, $T_1 = 1/\gamma_\parallel$, the same for both active and passive atoms which is consistent with the assumption of resonance between the emitting and absorbing transitions. We fix $\omega = 0.01$, i.e., we take the transverse relaxation time, $T_2 = 1/\gamma_\perp$, to be two orders of magnitude smaller than T_1. We also take $\rho \simeq 0.1$ and $\omega < \rho$. For related problems with, however, different parameter ranges see Knapp, Risken, and Wollmer.[5]

The pumping rates are hidden in A and C. Without absorber the threshold for laser behavior is at $A = 1$, and the steady-state laser intensity grows with $A - 1$. The absorbing medium is pumped as an emitter if C is smaller than unity, with threshold at $C = 1 - 1/r_1$. Thus we restrict C to be larger than unity.

For

$$(1-r_1)^{-1} \equiv C_{tc} \leq C \leq (\rho + r_1)/\rho(1-r_1) \equiv C_{osc},$$

the zero-field solution is stable up to $A = C$ where a branch of steady solutions bifurcates subcritically like in a first-order (hard mode) transition.[3] Thus there is a region of values of A where two steady states are available to the system, since the middle branch is always unstable (see Fig. 1). Hysteretic phenomena have been observed in experiments.[6] Note that in a related problem[7] oscillations exist for A larger than C_{osc} which corresponds to the standard Hopf bifurcation.[8] We do not consider this case here.

The linear stability analysis of the *upper nonlinear branch* of steady values shows that it is *unstable* for the range of parameters given earlier in this Letter. This phenomenon is mathematically analogous to a case found in optical bistability.[9] It shows the danger of a straightforward extension of the equilibrium phase transition picture to (dynamic) nonequilibrium problems (as done, for instance, in Refs. 6 and 10). Even though one is able to build a Landau *potential* which has as extrema the different steady states of (1) or (2) —or a similar system—knowing that a state is a minimum is not enough to assess its stability. One must show that the Landau potential is a Lyapunov functional of the system. This is not an easy task and it has been disregarded in all *oversimplified* descriptions of dynamic transitions.[11] In the laser without absorber the potential appears associated to a one-dimensional Fokker-Planck equation when there is adiabatic elimination of all variables other than the field.[12-14] It turns out that the smallness of ω with respect to ρ (which is of order or smaller than 0.1) forbids the elimination here.

TABLE I. Characteristics of the oscillations [pulse intensity, time period, and full width at half maximum (FWHM)] for $\rho = 0.1$, $r_1 = 0.4$, and $C = 6$. As a reference, the intensity of the (unstable) steady state (upper branch in Fig. 1) is 2.7 for $A = 6.1$. Units are in accordance with the scales introduced in the main text.

A	Intensity (max/min)	Period	FWHM
6.05	$26.0/4\times10^{-5}$	675	12
6.10	$26.0/4\times10^{-4}$	433	12
6.15	$25.5/2\times10^{-3}$	327	13
6.20	$23.5/1\times10^{-2}$	260	16

Returning to Fig. 1 we have also found that at $A = A_u$ there is a right-hand bifurcation from the *upper nonlinear branch.* With use of Floquet theory[8] and a two-time scale method[15] it appears that the bifurcation is subcritical to limit cycle and, therefore, initially unstable. The steady state is stable for $A \geq A_u$.

All these results are valid in a range of values around $\rho = 0.1$, and $r_1 < 1$ (for illustration we set $r_1 = 0.4$). The value of ρ corresponds to a 1-m cavity with some 3% losses if we take $\gamma_\perp \approx 10^{-8}$ s^{-1}. The condition $r_1 < 1$, and thus $\overline{T}_2 > T_2$, is consistent with having a gain cell with higher gas pressure than that in the absorber.[16]

The divergence of the vector field (2) is *always* negative and there must be an attractor in the solutions. As none of the steady states is stable nor does any smooth limit cycle bifurcate from

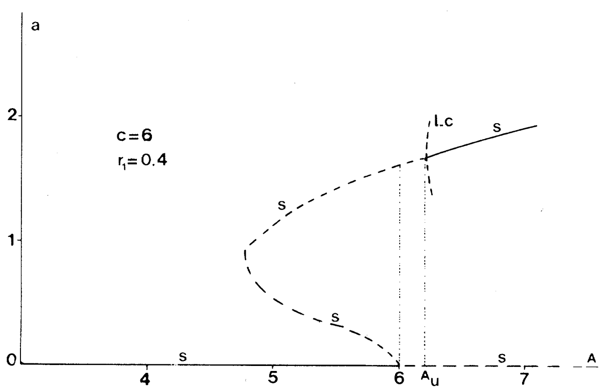

FIG. 1. Bifurcation diagram of (2) at $\rho = 0.1$, $r_1 = 0.4$, and $C = 6$. Solid and broken lines indicate *stable* and *unstable* solutions respectively. The limit cycle (LC) bifurcates *subcritically* at $A = A_u = 6.2344$. S denotes steady states. The vertical dotted lines delineate the region of Q switching ($C = 6 < A < A_u$). Units are in accordance with the scales introduced in the main text.

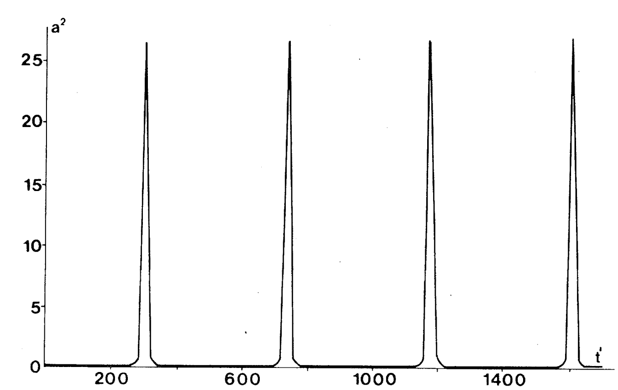

FIG. 2. Train of pulses (Q switching) as a particular case of limit-cycle solution of (2) for $\rho = 0.1$, $r_1 = 0.4$, $C = 6$, and $A = 6.10$. The electric field intensity is a^2. The time (t') unit is $T_2 = 1/\gamma_\perp$. The height of the pulses is $a_{max}^2 = 26$. Units are in accordance with the scales introduced in the main text.

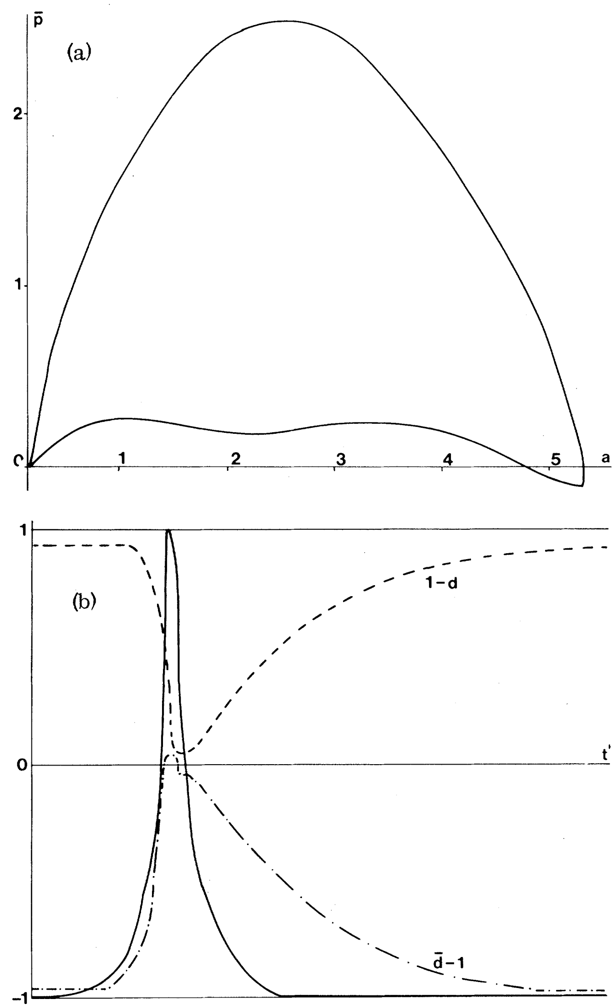

FIG. 3. The relevant quantities in the Q switching of (2). (a) Polarization of atoms in the absorber vs electric field amplitude for a time period in the limit cycle. Note the cusp-like approach to the origin of coordinates as expected in a saddle-loop singularity. (b) Time evolution, during a period, of population inversions (emitter, $1-d$; absorber, $\bar{d}-1$). For illustration, the solid line accounts for the pulse (not to scale here). Values of 1 (respectively, -1) account for all atoms in the excited (respectively, ground) state. Parameter values are those used in the preceding figures and units are in accordance with the scales introduced in the main text.

A_u to the left, we have explored the region $C<A<A_u$ by means of the Poincaré map with numerical integration of (2). We have located a relaxation oscillation as shown in Fig. 2. This limit cycle appears as a *finite-amplitude* solution and bears dramatic similarity with the experimentally observed self pulses in passive Q switching.[1,2,16] Table I gives an illustration of the results. The width of the pulses is of the order of the photon lifetime in the cavity, i.e., $1/\kappa$ ($1/\rho$, in dimensionless units). The period is of the order of the decay time of the excited state, T_1 ($1/\omega$ in our units).[17]

Figure 3 illustrates the behavior of the relevant quantities (polarization and population inversion) during one cycle. Note that at $\bar{d}=1$ the absorber becomes transparent with equal numbers of atoms in the excited and ground states. It actually becomes *active* ($\bar{d}>1$) for a short interval during the rising of the pulse, and *cooperates* with the active cell.

Table I also shows that the pulse peak intensity decreases with increasing pumping rate, A. Its value is an order of magnitude higher than the value of the corresponding *unstable* steady state. The time interval between pulses decays with increasing A. These features agree with the description given by Hanst, Morreal, and Henson.[1] When A tends to A_u from below, the pulses broaden and tend more and more towards a smooth limit cycle keeping, however, a nonzero amplitude and finite period at $A=A_u$. Thus, we expect this Q-switched branch to join the *initially* unstable solution (somewhere at $A>A_u$). When A approaches C from above, the limit-cycle period rises dramatically, the minimum intensity tends to zero, and the peak intensity remains essentially constant. These features are characteristic of a saddle loop[18] at $A=C$ [Fig. 3(a)]. At A around C the cycle tends to a single pulse with infinite rising and decaying times but, however, finite width and height, very much like in the π pulses described by Arecchi and Bonifacio.[19] It is not a hyperbolic-secant pulse since it does not rise from zero exponentially but according to some power law as the trivial zero state is only marginally unstable at $A=C$.

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