PHYSICAL REVIEW LETTERS

VOLUME 47 | 28 DECEMBER 1981 | NUMBER 26

Classical Derivation of the London Equations

W. Farrell Edwards
Physics Department, Utah State University, Logan, Utah 84322
(Received 15 June 1981)

The London equations, which were proposed as restrictions to conventional classical electromagnetism in order to explain the Meissner-Ochsenfeld effect in superconductors, are derived from familiar classical action integrals. The equations of motion, from which the London equations follow, should apply to some collisionless plasmas. Space and thermonuclear applications are timely and significant.

PACS numbers: 03.50.De, 11.10.Ef, 52.30.+r, 74.30.-e

Outside of zero resistance itself, the most characteristic phenomenon in superconductivity is the Meissner-Ochsenfeld effect wherein a magnetic field inside a resistive material is driven out, except for a small penetration depth, when the material becomes superconducting.[1] Conventional electromagnetism, which includes Maxwell's field equations and the Lorentz force relation, predicts that the field will be locked in.

In order to explain the Meissner effect, London and London[1] in 1935 proposed *ad hoc* restrictions on classical theory.[2] Now known as the London equations, these restrictions were later given a quantum-mechanical foundation, first in phenomenological theories[3,4] and finally in the microscopic BCS theory.[5]

The Meissner effect is of such significance that, in terms of it, a distinction is often made between superconductors and perfect conductors,[6] a perfect conductor being defined as one having zero resistance but obeying classical electromagnetism, whereas a superconductor is one having zero resistance but obeying quantum mechanics. The latter displays the Meissner effect, whereas, according to the argument, the former should not.

Until recently, experimental tests of the assertion concerning perfect conductors have not been possible because of the unavailability of classical systems having sufficiently low resistance, or, equivalently, long particle-collision times. Now, however, low-density plasmas in regions probed by space vehicles have particle-collision times of seconds and longer, and hot plasmas in thermonuclear devices have collision times in the millisecond range. Hence tests have become possible.

Furthermore, this paper reports a fundamental change in the theoretical picture: Despite the unsuccessful early efforts to explain the Meissner effect using classical electromagnetism, and despite present confidence that quantum mechanics is the only possible approach, the London equations do indeed have a classical derivation that applies to superconductors and to some collisionless plasmas as well.

Many attempts have been made to formulate action-based electrodynamics of a charged fluid using Eulerian variables. Eckart[7] outlined a procedure to use with field-theory Lagrangian densities and he proposed various *ad hoc* alterations to accepted, classical Lagrangian densities. Many authors[8] have used the approach to describe a charged fluid but have imposed Lin's constraint[9] or a similar restriction in order to get the correct momentum-moment equations for a fluid and in so doing eliminated the possibility of deriving the London equations.

Eulerian action integrals for collisionless fluids at absolute zero of temperature are easily formulated by transforming well-accepted, classical Lagrangian actions into Eulerian notation. After that the equations of motion are obtained by applying the principle of least action.

We must remember that implicit in the action-integral approach is the assumption that the total energy is constant. Only with considerable difficulty can actions be used with dissipative systems, so that the derivation is limited to perfect conductors, which for our purposes we define as fluids composed of particles having mean collision times that are long compared with the persistence times of the phenomena of interest. This definition will ensure that losses are negligible.

Field-theory action integrals.—We begin with a well-known, field-theory action integral,[10] which generates a complete, self-consistent set of system equations with fully interacting fields and source particles, the source current density being expressed in Lagrangian notation. It is

$$I_1=\int(F_{\beta\sigma}F^{\beta\sigma}/4\mu_0)d^4x+\iint\sum_i q_i\delta^4 u_{i\sigma}A^\sigma d\tau_i d^4x+\int\sum_i m_i c(u_{i\sigma}u_i{}^\sigma)^{1/2}d\tau_i\,, \tag{1}$$

where

$$F_{\beta\sigma}=\partial_\beta A_\sigma-\partial_\sigma A_\beta \tag{2}$$

is the electromagnetic field tensor, A_σ is the potential, and q_i, m_i, $u_i{}^\sigma$ $(=dx_i{}^\sigma/d\tau_i)$, and τ_i are the ith source particle's charge, mass, velocity and proper time, the latter being the independent variable for source quantitites expressed in Lagrangian notation. The Dirac delta function $\delta^4(x^\nu-x_i{}^\nu)$ connects x^ν, the independent variable for field quantities, with the four-position of the ith particle, $x_i{}^\nu(\tau_i)$. SI notation is used throughout. We employ the Minkowski metric where $x^\nu=(\mathbf{r},ct)$ and $g_{\beta\beta}=(-1,-1,-1,1)$.

If at this point we were to apply the principle of least action treating A^σ and $x_i{}^\sigma$ as having independent degrees of freedom, we would obtain, as equations of motion, Maxwell's field equations and the Lorentz force relation,

$$\partial^\beta F_{\beta\sigma}=\mu_0 j_\sigma, \tag{3}$$

$$m_i du_i{}^\beta/d\tau_i=q_i F_i{}^{\beta\sigma}u_{i\sigma}, \tag{4}$$

where the four-current density is given by

$$j^\sigma(x^\nu)=\sum_i q_i\int u_i{}^\sigma\delta^4 d\tau_i. \tag{5}$$

We wish to express the Eulerian equivalent to I_1. This is accomplished by transforming Eq. (1) with use of Eq. (5) which gives

$$I_2=\int\left(\frac{F_{\beta\sigma}F^{\beta\sigma}}{4\mu_0}+A_\beta j^\beta+\frac{mc}{q}(j_\beta j^\beta)^{1/2}\right)d^4x, \tag{6}$$

where we have assumed that the fluid has a constant ratio of rest-mass density to rest-charge density, $\rho_{m0}/\rho_0=m/q$. The Lagrangian density L_2 obtained from Eq. (6) and $I_2=\int L_2 d^4x$ is given in a slightly different form by Panofsky and Phillips.[11]

Equations of motion.—To derive the equations of motion we apply the Euler-Lagrange equations to L_2, treating A_β and j_β as having independent degrees of freedom. The variation with respect to A_β leads to Maxwell's equations as shown by Panofsky and Phillips,[11] who, however, do not complete the set of equations by taking the variation with respect to j_β.

For the j_β variation the Euler-Lagrange equations become simply $\partial L_2/\partial j_\beta=0$, which leads easily to

$$A_\sigma+(m/q)u_\sigma=0, \tag{7}$$

where we have used $j_\beta j^\beta=(\rho_0 c)^2$ and $j_\sigma=\rho_0 u_\sigma$.

Equation (7) is very important in superconductivity theory and is usually derived from quantum mechanics. It forms the crux of the present thesis, and its easy derivation from a well-accepted, classical action, Eq. (6), establishes the discovery that (a) superconductivity phenomena following from Eq. (7) are classical in nature, and (b) other fluid-type systems, including plasmas that satisfy the conditions controlling the use of I_2, are also governed by Eq. (7) and hence should exhibit the same superconductivity-type phenomena.

Some properties of Eq. (7).—The Eulerian form of the Lorentz force relation,

$$\rho_{m0}u^\alpha\partial_\alpha u^\sigma=F^{\sigma\beta}j_\beta, \tag{8}$$

can be derived from Eq. (7) by use of Eq. (2) and $u^\beta\partial_\sigma u_\beta=\frac{1}{2}\partial^\sigma(u_\beta u^\beta)=0$.

On the other hand, if we attempt to derive Eq. (7) from Eq. (8) we get only as far as $u_\beta\{\partial^\sigma[A^\beta-(m/q)u^\beta]-\partial^\beta[A^\sigma-(m/q)u^\sigma]\}=0$. From this it is clear that, although solutions to Eq. (7) also solve Eq. (8), the converse is not necessarily true. Of course, most solutions satisfy both Eqs. (7) and (8), plasma oscillations for example, but the few that do not, such as the Meissner effect, indicate a fundamental, physically observable difference between the equations.

The gauge in which A^σ is expressed is not the usual Lorentz gauge, $\partial_\sigma A'^\sigma = 0$. Nor is it the London gauge, $\nabla\cdot\vec{A} = 0$, although it includes the latter as a special case when ρ is uniform. The gauge is characterized by

$$\partial_\sigma(\rho_0 A^\sigma) = 0, \tag{9}$$

which is evident from Eq. (7), from which we also note that $A_\sigma A^\sigma$ = a space-time constant.

To transform Eqs. (3) and (7) to the Lorentz gauge we use the gauge transformation $A_\sigma = A'_\sigma + \partial_\sigma\lambda$ and then impose $\partial^\sigma A'_\sigma = 0$ which determines λ. (Note that $\partial_\sigma\partial^\sigma\lambda \neq 0$.) This results in modified equations of motion which include

$$A'_\sigma + \partial_\sigma\lambda + (m/q)u_\sigma = 0, \tag{10}$$

$$\partial_\beta\partial^\beta A'_\sigma = \mu_0 j_\sigma, \tag{11}$$

and the Lorentz-gauge condition.

Derivation of the London equations.—We derive the London equations in their covariant form simply by using Eq. (7) to eliminate A_σ from (2) which results in

$$F_{\beta\sigma} = (m/q)(\partial_\sigma u_\beta - \partial_\beta u_\sigma), \tag{12}$$

the three-dimensional, low-velocity form of which is $\vec{B} = -(m/q)\nabla\times\vec{u}$ and $\vec{E} = (m/q)(\partial\vec{u}/\partial t + \frac{1}{2}\vec{\nabla}u^2)$. Thus, the London equations, and consequently the Meissner-Oschenfeld effect, have a classical derivation.

Although one cannot claim that perfect conductors and superconductors are identical (there are many superconductor effects, including the occurrence of zero resistance in a solid material, that result from the electron-lattice interaction and are clearly quantum mechanical), we see that they share many electromagnetic properties including the Meissner effect. In fact, these very properties, even in superconductors, are classically guaranteed once zero resistance is shown to arise.

To ensure that the equations of motion are not an artifact of the use of a particular theoretical approach, I have derived them from direct-action theory. When the Schwarzschild-Tetrode-Fokker action, which was used by Wheeler and Feynman[12] in their interparticle, direct-action, electromagnetic theory, is transformed from Lagrangian to Eulerian notation and the dependent variables are varied according to the principle of least action, the same Lorentz-gauge form of the equations of motion, Eqs. (11) and (12), result as in the field-theory approach. The derivation will be reported elsewhere.

Kinetic effects and external fields.—The formulation is not limited to fluids at absolute zero. Certain kinetic effects can be incorporated into the theory as long as the conditions to be discussed in the next section are satisfied. We can, at the same time, extend the equations to include external fields produced by constrained currents.

For a warm plasma having N free species, and external fields arising from M external, constrained currents, the low-velocity form of Eq. (10) becomes

$$\vec{A} + (m_\eta/q_\eta)\vec{v}_\eta - \nabla\lambda_\eta = 0, \tag{13}$$

$$\varphi + \frac{m_\eta}{2q_\eta}v_\eta^{\,2} + \frac{3k}{2q_\eta}T_\eta + \frac{\partial\lambda_\eta}{\partial t} = 0, \tag{14}$$

where the charge, mass, temperature, and macroscopic velocity of species η are denoted by q_η, m_η, T_η, and $\vec{v}_\eta$. There are N sets of equations. $\vec{A}$ and φ are the total fields produced by currents $j^\sigma = \sum j_i^{\,\sigma}$, where i extends to $N+M$. Equation (11) holds for the total fields. Note that, in going from microscopic to macroscopic variables, the form remains the same except for the introduction of the temperature term.

With use of Eqs. (13) and (14) the collisionless form of the momentum moment equation can be derived; hence Eqs. (13) and (14) restrict the solutions of the momentum moment equation for a collisionless plasma just as (7) restricts the solutions of the Lorentz-force relation, Eq. (8). The derivation of Eqs. (13) and (14) will be reported elsewhere.

Free energy and other conditions.—As discussed earlier, the primary condition on the use of Eq. (7) is that the fluid is nondissipative (collisionless, in the case of plasmas), but there are other conditions as well. In superconductors it is well known that above a critical value B_c, an applied magnetic field causes the material to go normal (resistive) and Eq. (7) no longer applies. This occurs when the available free energy is less than the magnetic field energy and consequently the field cannot be driven out as required by the steady-state Meissner effect.

This condition is necessary (although not sufficient) also for plasmas.[13] The dominant term in the free-energy density for many plasmas is nkT; hence, the free-energy condition becomes

$\beta > 1$ where $\beta \equiv nkT/(B^2/2\mu_0)$. Thus, given appropriate boundary and initial conditions, high-β plasmas should exhibit the Meissner effect for a time on the order of the collision time, whereas low-β plasmas should show the familiar frozen-in field effect.

We must also insist that treating the ensemble of particles as a fluid is a good approximation. Effects causing the particles to move individually rather than collectively could invalidate Eq. (7).

Classical tests and applications.—Magnetic flux ropes in the Venerian ionosphere[14,15] bear a strong resemblance to fluxoids in type-II superconductors.[16] The London penetration depth, $\lambda = (m/\mu_0 nq^2)^{1/2}$, which is a reasonable estimate of the radius of fluxoids, varies with altitude from 3.8 to 7.7 km, in good numerical and functional agreement with the 7.3- to 16-km variation observed for the radii by the Pioneer-Venus spacecraft.

The O^+ ion has a sufficiently long collision time ($\tau_c \sim 20$ s), whereas the associated electrons are collision dominated ($\tau_c \sim 0.08$ s) which is also necessary for the flux-rope solution. The free-energy parameter β hovers near unity. In further confirmation of the theory, perfect diamagnetism and flux ropes appear when $\beta > 1$ (observed $\frac{2}{3}$ of the time), and magnetic penetration occurs otherwise. Appropriate boundary conditions are provided by the Venus bow shock and ionopause. The ionosphere is perturbed from the frozen-in field condition when β rises above unity. This imposes Eq. (7) for approximately one collision time, followed by a slower field diffusion back into the region. A detailed report on the Venus flux ropes is in preparation.[17]

The combination of conditions near Venus appears to be almost unique in the solar system although under special perturbing circumstances similar unconventional phenomena might be observed elsewhere, especially near the $\beta = 1$ boundary surface of the solar wind plasma.

Applications in thermonuclear plasma research are likely and preliminary studies show promise. In hot deuterium plasmas, the deuteron collision time approaches the containment time; consequently, conventional solutions for high-β plasmas should be restricted. In fact, when we consider the high degree of stability of superconductor fluxoids we are led to inquire whether or not the formalism might generate new stability criteria for thermonuclear reactors.

I am greatly indebted to my friends and colleagues R. Allen, D. A. Baker, P. Banks, A. P. Edwards, B. F. Edwards, B. C. Jensen, V. G. Lind, and R. C. Thompson for helpful discussions and other support. Funding was received from Research Corporation and the Utah State University Research Council.

---

[1]W. Meissner and R. Ochsenfeld, Naturwissenschaften 21, 787 (1933).

[2]F. London and H. London, Proc. Roy. Soc. London, Ser. A 149, 71 (1935).

[3]A. B. Pippard, Proc. Roy. Soc. London, Ser. A 216, 547 (1953).

[4]V. L. Ginzburg and L. D. Landau, Zh. Eksp. Teor. Fiz. 20, 1064 (1950).

[5]J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 108, 1175 (1957).

[6]M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1975), p. 2.

[7]C. Eckart, Phys. Rev. 54, 920 (1938).

[8]See, for example, C. H. Su, Phys. Fluids 4, 1376 (1961); P. Penfield, Jr., and H. A. Haus, Phys. Fluids 9, 1195 (1966).

[9]J. Serrin, in *Handbuch der Physik* (Springer-Verlag, Berlin, 1959), Vol. VIII, pp. 144–150.

[10]A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Macmillan, New York, 1964), p. 140.

[11]W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed. (Addison-Wesley, Reading, Mass., 1962), p. 452.

[12]J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. 21, 425 (1949).

[13]This condition was developed by B. F. Edwards and the author.

[14]C. T. Russell and R. C. Elphic, Nature (London) 279, 616 (1979).

[15]R. C. Elphic *et al.*, J. Geophys. Res. 85, 7679 (1980).

[16]P. M. Banks, W. F. Edwards, C. Rasmussen, and R. C. Thompson, Geophys. Res. Lett. 8, 95 (1981).

[17]B. F. Edwards, W. F. Edwards, and R. C. Thompson, unpublished.