Critical Dynamics of an Interface in $1+\epsilon$ Dimensions

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A model describing the relaxation dynamics of an interface of Ising-like systems is introduced. By means of renormalized field theory in $d=1+\epsilon$ dimensions the dynamic critical exponent is found as $z=2+\epsilon-\frac{1}{2}\epsilon^2+O(\epsilon^3)$. Interpolation with the known result near four dimensions yields good agreement with a high-temperature expansion and with recent real-space and Monte Carlo renormalization-group calculations in two dimensions.

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The interest in the study of dynamic critical phenomena over the past two decades has been focused primarily on bulk properties.[1] Comparatively little is known about the critical dynamics of surfaces or interfaces.[2] In particular no model has been developed so far which permits a systematic analytic investigation of time-dependent fluctuations of an interface. In this Letter we propose such a model for the interface of an Ising-like system whose dynamic critical behavior can be studied by means of renormalized field theory near the lower critical dimension[3] $d_c=1$.

This is a purely relaxational model which constitutes the simplest dynamical generalization of the static version recently introduced.[4] It provides a new, systematic approach to the unresolved[5] problem of predicting the dynamic critical exponent z for the $d=2$ kinetic Ising model[6] or for the continuum version,[7] model A. Our (two loop) expression for z is $z=2+\epsilon-\epsilon^2/2+O(\epsilon^3)$, where $\epsilon\equiv d-1$.[8] A Padé interpolation between $d=1$ and $d=4$ using the two-loop result from the four-dimensional expansion leads to good agreement with the high-temperature expansion of Rácz and Collins[5] as well as with recent real-space and Monte Carlo renormalization-group calculations by Achiam,[5] by Mazenko *et al.*,[5] and by Tobochnik *et al.*[5] in $d=2$ dimensions. An extension of our approach to three-loop order appears feasible. Since z is known to three loops[9] near $d=4$, it may be interesting to compare the three-loop interpolation with other results. Further, our methods may be generalized to study the dynamic universality classes of models[7] B, C, and D. In particular, an analysis of the interface of liquids [model H (Ref. 10)] in $1+\epsilon$ dimensions would be of great interest but requires the inclusion of reversible flow terms.

Following Wallace and Zia[4] (WZ), we consider a $(d-1)$-dimensional interface imbedded in a d-dimensional bulk system. The *static* behavior of such a model is described by the partition function $\int Df\exp(-\mathcal{H}/T)$ with

$$\mathcal{H}=\mathcal{H}_0+\tfrac{1}{2}\int d^{d-1}x\, m^2 f^2;\quad \mathcal{H}_0=\int d^{d-1}x\sqrt{g}, \tag{1}$$

where $g\equiv 1+(\nabla f)^2$. Here, $f(\vec{x})$ is a local height variable[11] describing the fluctuations of the position of the interface at the point $\vec{x}$ relative to a reference plane $f(\vec{x})=0$. This plane is defined via the pinning potential $m^2f^2/2$, which is necessary to stabilize the interface for $d\leq 3$. The $\mathcal{H}_0$ term, being proportional to the total area of the interface (without overhangs), is just the surface tension energy. WZ showed that it is renormalizable near $d=1$. Thus a systematic expansion in ϵ exists for describing the static critical properties.

A proper description for the *dynamics* of the fluctuating variable $f(\vec{x},t)$ is not obvious, even for the simplest case of pure relaxation for a system

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with a nonconserved order parameter. For example, one might expect an equation of motion of the form $\dot{f} \equiv \partial f/\partial t = -\lambda \partial \mathcal{H}/\delta f + \zeta$, with $\lambda > 0$ being the Onsager coefficient and ζ a random force. From discussions below, it will be clear that there are difficulties with such a model. Instead, we propose the equation of motion

$$\dot{f} = -\lambda g^{1/2} \delta\mathcal{H}/\delta f + \zeta, \tag{2}$$

$$\langle \zeta(\vec{x},t)\zeta(\vec{x}',t')\rangle = 2\lambda g^{1/2}\delta(\vec{x}-\vec{x}')\delta(t-t'), \tag{3}$$

with a Gaussian random force. Because of the f-dependent factor $\sqrt{g}$, Eqs. (2) and (3) should be complemented by an interpretation rule which guarantees relaxation towards the correct equilibrium distribution $\exp(-H/T)$. This, however, turns out to be automatically fulfilled within the dimensional regularization scheme employed in the field-theoretic treatment below.

We briefly discuss three main salient aspects of our model: (i) physical interpretation, (ii) Euclidean covariance, and (iii) relation with model A.

(i) The physical picture associated with (2) is especially transparent if one recognizes that $\dot{f}/\sqrt{g}$ is a velocity perpendicular to the interface. This is so since $\delta f/\sqrt{g}$ is the normal displacement of the interface given an infinitesimal change in f (at $\vec{x}$) to be δf. On the other hand, $\delta\mathcal{H}_0/\delta f$ is the increase in energy associated with the change in area of the surface element at $\vec{x}$. $\delta\mathcal{H}_0/\delta f$ may also be related to the (extrinsic) curvature tensor

$$K_{ij} = [\nabla_i\nabla_j f - (\nabla_i f \nabla_k \nabla_j \nabla_k f) g^{-1}]/\sqrt{g},$$

where $i,j,k = 1, \ldots d-1$. One verifies that $\delta\mathcal{H}_0/\delta f = -\mathrm{Tr}K$ which leads to the physically reasonable picture of the total (local) curvature being the hydrodynamic driving force.

(ii) Part of the full d-dimensional Euclidean symmetry of the Ising-like bulk system is nonlinearly realized[4,12] in the static model (with $m^2 = 0$). In particular, consider a z-x_i rotation by an infinitesimal angle θ_i. Denoting the infinitesimal change in a quantity Q by $\theta_i \delta_i Q$ we have $\delta_i f = x_i + f\nabla_i f$ so that $\delta_i \dot{f} = \nabla_i(f\dot{f})$ and $\delta_i\sqrt{g} = \nabla_i(f\sqrt{g})$. Further, since $\mathcal{H}_0$ is an invariant functional of $(\nabla f)^2$, we have $\delta_i(\delta\mathcal{H}_0/\delta f) = f\nabla_i(\delta\mathcal{H}_0/\delta f)$. Thus, the naive guess of $\dot{f} = -\lambda\delta\mathcal{H}_0/\delta f$ is not even a covariant equation. On the other hand, $g^{1/2}\delta\mathcal{H}_0/\delta f$ indeed transforms as $\dot{f}$, verifying that the systematic part of (2), for $m^2 = 0$, respects the full Euclidean symmetry as expected from the physical interpretation given above. The covariance of our model, *including* the random part ζ, is most efficiently discussed in terms of the dynamic funtional[13]

$$J[f,\tilde{f}]/T = \int dt\, d^{d-1}x[-\lambda g^{1/2}\tilde{f}^2 + \tilde{f}(\dot{f} + \lambda g^{1/2}\delta\mathcal{H}/\delta f)]/T, \tag{4}$$

where $\tilde{f}$ is a response field.[14] This J (with $m^2 = 0$) is invariant since $\delta_i \tilde{f} = f\nabla_i \tilde{f}$ as follows from demanding the invariance of $\int \tilde{f}\dot{f}$. Conversely, Euclidean invariance is so strong that the form (4) for J may be determined *uniquely* within the dimensional regularization scheme. Specifically we impose on J the following: (a) detailed balance with $\mathcal{H}$ being the static Hamiltonian, (b) locality, (c) presence of $(\nabla f)^2$ terms *only*, a necessary condition for renormalizability, and (d) invariance under the transformations specified above. The first condition implies that the coefficients of $\tilde{f}^2$ and $\tilde{f}\,\delta\mathcal{H}/\delta f$ are equal and opposite.[15] The uniqueness of J under these constraints is paralleled by the uniqueness of $\mathcal{H}_0$ in statics. Since this invariance of $\mathcal{H}_0$ is responsible for the renormalizability of the static theory, we believe that it will also ensure the renormalizability of our model.

(iii) Another appealing feature of this model is that it can be obtained directly from the ordinary model A,

$$\dot{\varphi} = -\Gamma\,\delta F/\delta\varphi + \theta,$$
$$F = \tfrac{1}{2}\int d^d x[(\nabla\varphi)^2 + \mu^2(\varphi^2 - 1)^2], \tag{5}$$

with a one-component order parameter φ and a random force θ, by taking the low-temperature limit $\mu^2 \to \infty$, in a manner similar to Diehl, Kroll, and Wagner.[16] Unlike their approach, we employ a saddle-point method, avoiding the lengthy process of resumming a perturbation series. The time-dependent $\varphi(\vec{x}, z, t)$ is decomposed into two parts:

$$\varphi(\vec{x}, z, t) = \chi(z - f(\vec{x}, t)) + \eta(\vec{x}, z, t), \tag{6}$$

where $\chi(\xi) = \tanh(\mu\xi/\sqrt{g})$ is the kink solution[16] which minimizes F. If we take η to be $O(1/\mu)$, a consistent, systematic expansion in powers of $1/\mu$ follows. Note that the only time dependence in the first term of (6) comes through $f(\vec{x}, t)$, corresponding to the statement that, as $T \to 0$ ($\mu \to \infty$), the most important dynamical variable is f. All degrees of freedom other than f can be projected out by the scalar product $(\chi_z, A) \equiv \int_{-\infty}^{\infty} dz(\partial\chi/\partial z)^* A$ applied to (5) with φ replaced by (6). This leads to Eqs. (2) and (3) with $\Gamma = \lambda$. Another way is to begin with the dynamic functional for model A, $J[\varphi, \tilde{\varphi}]$, and to take the limit

$\mu \to \infty$. This yields (4) with $\tilde f$ being proportional to $(\chi_z, \tilde\varphi)$. It is reassuring that different approaches lead to the same model.

Given J, Eq. (4), the program for studying dynamical critical behavior via the renormalization group is standard.[17] Apart from the renormalization of the static[4] bare quantities (denoted by a subscript B from now on) T_B, m_B, and f_B, we need to renormalize the dynamic ones λ_B and $\tilde f_B$:

$$T_B = \kappa^{-\epsilon} Z_T T, \quad m_B^2 = Z_T m^2, \quad f_B = f,$$
$$\lambda_B = Z_T^{-1} Z_\lambda \lambda, \quad \tilde f_B = Z_T \tilde f, \tag{7}$$

where T is now the dimensionless renormalized coupling.

Unlike for model A, Z_λ is nontrivial resulting from the extra factor $\sqrt{g}$ in Eq. (2). The dissipation-fluctuation theorem no longer determines Z_λ in terms of the renormalization of $\tilde f$. As a result of pure gradient couplings, $\tilde f$ is renormalized simply by Z_T. Using minimal renormalization we find to two-loop order $Z_\lambda = 1 + T^2/4\epsilon + O(T^3)$. Together with the static result[4] $Z_T^{-1} = 1 - T/\epsilon - T^2/4\epsilon$ this yields the function $\zeta(T) \equiv (\kappa\partial_\kappa \ln\lambda)_0 = T + O(T^3)$. It enters in the renormalization-group equation

$$[\kappa\partial_\kappa + \beta\partial_T + \gamma_1(m^2\partial_{m^2} - \tilde N) + \zeta\lambda\partial_\lambda]\Gamma_{\tilde N N} = 0 \tag{8}$$

for the vertex functions $\Gamma_{\tilde N N}$ ($\tilde N$ and N denote the number of external $\tilde f$ and f legs of the corresponding diagrams). The functions $\beta(T)$ and $\gamma_1(T)$ are given in statics.[4] At the critical fixed point $T^* = \epsilon - \epsilon^2/2 + O(\epsilon^3)$ we thus find the dynamic critical exponent $z = 2 + \zeta(T^*) = 2 + \epsilon - \epsilon^2/2 + O(\epsilon^3)$ appearing in the characteristic frequency $\omega_c(k, \xi) = k^z \Omega(k\xi)$. The latter determines the wave-vector and temperature dependence of the relaxation rate of fluctuations of the interface variable f. Although there is no rigorous proof that z is also the bulk dynamic exponent, we would argue along the lines of WZ that, as $\epsilon \to 0$ and $T_c \to 0$, the only important fluctuations of the bulk system come from the soft Goldstone modes. Furthermore, the close relation between our model and model A established above supports our conjecture that z can be identified with the bulk exponent of model A.

Setting $\epsilon = 1$ in our expression for z yields $z = 2.5$ in $d = 2$ dimensions. A more reliable value is obtained by a Padé estimate for z which interpolates between our result near $d = 1$ and the two-loop result near $d = 4$. The latter is $2 + c(4-d)^2 + O((4-d)^3)$; $c = [6\ln(\frac{4}{3}) - 1]/54 \cong 0.013$. The interpolation gives (see Fig. 1)

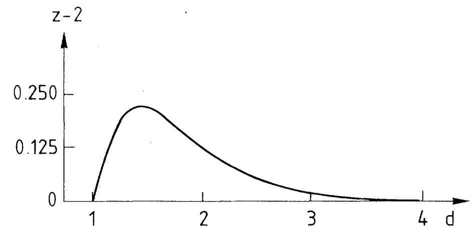

FIG. 1. The critical exponent $z - 2$ as a function of dimensionality d according to Eq. (9).

$$z - 2 = \frac{6c(4-d)^2(d-1)}{(2+60c) - (4+3c)d + (2-3c)d^2}. \tag{9}$$

Setting $d = 2$ in this formula yields $z \simeq 2.126$, which is in deceptively good agreement with the high-temperature series value of Rácz and Collins.[5] This also agrees well with real-space and Monte Carlo renormalization-group calculations.[5]

In conclusion, we wish to emphasize that our model for the dynamical behavior of an interface has sound physical interpretation, respects the full d-dimensional symmetry, and is intimately related to model A. Work on the generalizations to other models is in progress.

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Effective Harmonic-Fluid Approach to Low-Energy Properties of One-Dimensional Quantum Fluids

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A universal description of the low-energy properties of one-dimensional quantum fluids, based on a harmonic theory of long-wavelength density fluctuations with use of renormalized parameters, is outlined. The structure of long-distance correlations of a spinless fluid is obtained, showing the essential similarity of one-dimensional Bose and Fermi fluids. The results are illustrated by application to the one-dimensional Bose fluid with δ-function interaction.

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A recent study[1,2] of the one-dimensional (1D) Fermi fluid led to a simple low-energy description of it as a "Luttinger liquid": The low-energy effective Hamiltonian could be based on the spectrum of the Luttinger[3] model (which has a noninteracting elementary excitation spectrum of harmonic density fluctuations), plus residual anharmonic couplings that vanished at low energies; the structure of the theory was reminiscent of Fermi-liquid theory. In this Letter I report that the concept of an effective harmonic-fluid description applies quite generally to 1D quantum fluids independent of statistics, and the structure of their correlations becomes clear once a representation of the density operator has been constructed to reflect correctly the discrete particle nature of the fluid. Planar spin chains with axial symmetry can also be understood as a Bose fluid of "magnon" excitations about a fully aligned state. The theory of 1D quantum fluids can be applied to extract information from the solutions of exactly solvable (but opaque) models such as the Bose gas with finite-strength δ-function interaction.[4]

The Fermi fluid results obtained in Ref. 2 through Luttinger model techniques can be extended to the Bose fluid by considering the spin-$\frac{1}{2}$ Luttinger model with attractive $2k_F$ scattering

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