Gauge-Invariant Renormalization-Group Transformation without Gauge Fixing

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A generally applicable gauge-invariant renormalization transformation is proposed for the study of lattice gauge theories. Practical calculations can be carried out with use of Monte Carlo computer simulations. The power and efficiency of this approach is demonstrated by explicit calculations for the $Z(2)$ lattice gauge theory in three dimensions.

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Recently, Monte Carlo (MC) computer simulations[1-7] have been shown to be extremely useful in supplying information on the behavior of lattice gauge models,[8,9] which are important in several areas of physics. The study of phase transitions (or their absence) is essential to the understanding of quark confinement and asymptotic freedom in non-Abelian gauge theories of hadron dynamics.[1-7,9-13] There is also a very close relationship with problems involving random interactions in many-body theory, and the Hamiltonian (action) of the $Z(2)$ lattice gauge theory discussed below is just the frustration operator introduced by Toulouse as an essential tool in the study of spin-glasses.[14,15]

Both conceptually and for the efficiency of practical calculations, it is desirable to be able to analyze MC simulations of these models with the real-space renormalization-group (RG) formalism.[16-25] With the recent development of methods to determine the existence of phase transitions and calculate their locations accurately by means of MC simulations on small lattices,[13,23,24] the MCRG approach provides substantial improvement in the ability to extract information from computer simulations.

The use of real-space RG has proven more difficult than for the corresponding spin systems because of the high symmetry; most simple definitions of "block spins" fail to preserve the gauge symmetry and can only be used with some form of gauge fixing. However, for MC simulations of lattice gauge theories, gauge fixing is undesirable. It produces prohibitively long relaxation times when built into the MC simulation and it is a time-consuming procedure when employed in the definition of the renormalized operators.[13]

In this paper, I would like to propose a simple, generally applicable renormalization transformation for analyzing lattice gauge theories. The transformation avoids gauge fixing, either globally or within specified blocks,[13] and is directly applicable to arbitrary lattice gauge theories. The renormalized Hamiltonians (actions) retain the full gauge symmetry of the original model.

To demonstrate the practicality of this approach, I have performed explicit calculations on the nontrivial problem of the $Z(2)$ lattice gauge theory in three dimensions. This model was originally shown by Wegner[8] to be the dual of the three-dimensional Ising model. It is especially suitable for testing a new approach because it has a phase transition and the critical coupling constant and critical exponents are known to sufficient accuracy.[26,27]

Consider a general Hamiltonian of the form

$$H = K\sum_{P} \mathrm{Tr}(U_{ij}U_{jk}U_{kl}U_{li}), \tag{1}$$

where the sum is over elementary plaquettes, P, and the U_{ij}'s are operators associated with the links between lattice sites. These operators are elements of a group G and are defined such that $U_{ij} = (U_{ji})^{-1}$. [For the example of the $Z(2)$ lattice gauge theory, G contains only the scalars $+1$ and -1.]

These Hamiltonians are invariant under an arbitrary gauge transformation of the form

$$U_{ij} \Rightarrow U_{ij}' = G_i^{-1} U_{ij} G_j \,, \tag{2}$$

where the G_i's are members of G defined at each site. It is this symmetry that causes difficulties when an attempt is made to define "block spins" in analogy with the procedures that have proven effective in spin systems.

The renormalization-group transformation I am proposing is illustrated for simplicity on a two-dimensional lattice in Fig. 1. The lines represent the gauge variables, U_{ij}, and the intersections are the sites of the original lattice. For a transformation with scale factor $b = 2$, only the circled sites remain in the new renormalized lattice. The renormalized gauge fields connecting two sites of the renormalized lattice are constructed from the operator products along the paths marked A, B, and C as shown in Fig. 1. The products of

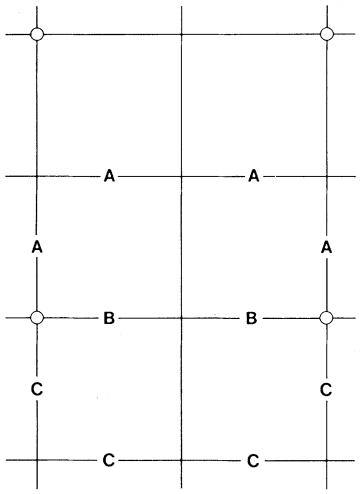

FIG. 1. Diagram of the proposed renormalization transformation for lattice gauge theories with $b = 2$ using $d = 2$ as an example. Operator products are formed along each of the paths marked A, B, and C. In d dimensions, there are $2d+1$ such paths for each pair of neighboring sites on the renormalized lattice.

the gauge fields over each of these paths contribute equally and the renormalized operator is determined by majority rule [as in the calculation described below for the $Z(2)$ model], or some convenient definition of a similar nature. The rule should be chosen to have the symmetry of the model, so that multiplying each of the operator products by the same element of G is equivalent to multiplying the renormalized operator by that element.

This transformation satisfies all symmetry requirements. Gauge transformations on sites retained by the RG transformation produce corresponding gauge transformations of the renormalized Hamiltonian. The renormalized operators are invariant under gauge transformations associated with any other site.

Table I gives the technical data for the MC simulations of the three-dimensional $Z(2)$ lattice gauge theory. They were performed at the value $K_c = 0.761\,336$, obtained from the series estimate for the critical coupling in the three-dimensional Ising model,[22] using Wegner's duality relation.[8] Table II shows the deviation of this coupling from the critical coupling calculated with use of comparisons of MC simulations on pairs of lattices following Wilson's technique as applied by Landau and Swendsen[13,23,24] for locating tricritical points. As the sizes of the lattices increase, more RG transformations are possible, reducing the size effect and the effect of irrelevant operators. However, because of critical slowing down, relaxation times increase, giving larger statistical errors. Both effects can be seen in the data in Table II. It is clearly quite simple to determine the existence of the phase transition and confirm the value of the critical coupling to an accuracy of better than 1%—even restricting consideration to the data from $4\times4\times4$ and $8\times8\times8$ lattices.

The eigenvalues $\lambda = b^y$ of the linearized RG transformation

$$T_{\alpha\beta} = \delta K_\alpha^{(n+1)} / \delta K_\beta^{(n)} \tag{3}$$

calculated at the fixed-point Hamiltonian H^* give the critical eigenvalue exponents.[17,20,21] As with other MCRG calculations, the matrix $T_{\alpha\beta}$ is found numerically by solving a set of linear equations with coefficients obtained from correlation functions calculated from MC simulations.[20,21]

The magnetic eigenvalue exponent, y_h, was found to be negative as expected. Table III shows the MCRG estimates for the critical eigenvalue exponent $y_T = 1/\nu$. The size effect is clearly

TABLE I. MC simulation data for MCRG calculations for the $Z(2)$ lattice gauge model. Data taken for correlation functions every 5 MC steps/site.

Lattice size (L)	Length of MC simulation	Steps discarded to reach equilibrium
32	38 800	15 000
16	105 000	40 000
8	580 000	80 000
4	1 040 000	60 000

TABLE II. $Z(2)$ lattice gauge theory. Estimates of the deviation of the coupling constant used in the MC simulation [$K_c = 0.761\,13$ from series expansions and duality (Ref. 26)] from the true value of K_c. The numbers in parentheses are estimates of the statistical error in the last digit. N_r is the number of RG iterations; N_c, the number of coupling constants in the RG analysis.

		Lattice sizes being compared		
N_r	N_c	32–16	16–8	8–4
1	1	0.0013(4)	0.0018(3)	0.0022(2)
	2	0.0012(8)	0.0015(5)	0.0018(4)
2	1	0.0013(7)	0.0020(4)	0.0035(5)
	2	0.0013(8)	0.0020(6)	0.0031(5)
3	1	0.0009(6)	0.001(1)	
	2	0.0009(6)	0.001(1)	

TABLE III. Estimates of the critical eigenvalue exponent y_T [actual value is about 1.57 to 1.60 (Refs. 26 and 27)] for the $d = 3$, $Z(2)$ lattice gauge theory as a function of the number of RG iterations (N_r), the number of coupling constants in the RG analysis (N_c), and the linear dimension of the lattice (L).

		Lattice size (L)			
N_r	N_c	32	16	8	4
1	1	1.37(2)	1.35(1)	1.32(1)	1.29(3)
	2	1.37(3)	1.35(1)	1.32(1)	1.10(9)
	3	1.36(3)	1.33(1)	1.27(1)	0.8(1)
2	1	1.47(3)	1.40(2)	1.30(1)	
	2	1.47(3)	1.40(2)	1.12(2)	
	3	1.45(3)	1.34(3)	0.70(3)	
3	1	1.49(5)	1.3(1)		
	2	1.48(5)	1.1(1)		
	3	1.41(7)	0.7(3)		

visible, but the values for the $32\times 32\times 32$ lattice suggest convergence towards the expected value for y_T. The accuracy is not as good as the corresponding MCRG calculation in the Ising spin representation, but it is still within 10% of the expected value at the second iteration. This provides a clear distinction between this second-order transition and a first-order transition that would have $y_T = d = 3$.[28] Specially optimized programs should be able to improve the accuracy considerably.[1] It should be mentioned that these estimates for the critical exponent are "biased," so that averages over many short runs can depend on the length of the individual runs. This dependence vanishes in the limit of long runs and is usually a very small effect in MCRG calculations. For the present calculation, the effect was significant for the $32\times32\times32$ lattice due to the long correlation times, even for individual runs as long as 6000 MC steps/site. The effect was not a problem with the smaller lattices. Since the correlation functions that are compared for the determination of the critical coupling are unbiased, the effect was not a serious problem there either.

In addition to testing this approach by direct application to problems of interest (especially for $d = 4$), it should be possible to investigate the properties of the transformation in the "weak-coupling" (large-K) limit in perturbation theory.[29]

The RG transformation I have proposed can be extended to include Higgs variables associated with the sites of the lattice and interacting with the gauge fields, by performing a gauge transformation on each site to rotate the Higgs field into the identity operator. The full effect of the Higgs fields is moved to the gauge operators, which can then be renormalized as described above.

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