Solvable Fractal Family, and Its Possible Relation to the Backbone at Percolation

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A nontrivial family of d-dimensional scale-invariant fractal lattices is described, on which statistical mechanics and conductivity problems are exactly solvable for every d. These fractals are finitely ramified but not quasi one dimensional, and hence can be used to model the important geometrical features of the percolating cluster's backbone. Critical exponents calculated for this model agree with those of "real" systems at low dimensionalities.

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Few nontrivial models in statistical mechanics are exactly solvable. This Letter shows that a new collection of self-similar models, made up of the Sierpiński gasket[1,2] and of its generalizations to all Euclidean dimensionalities (d), should be added to this short list. This is the first time that a family of nontrivial fractals[1,3] is studied both *systematically* and *exactly* for every d, and the corresponding transport properties are calculated exactly.[4] In addition to providing a testing ground for approximations, these fractal models may yield insight on the geometric structure of the percolation backbone.

Much of the current interest in the properties of dilute systems concentrates on the vicinity of the percolation threshold, p_c.[5,6] Monte Carlo (MC) simulations,[6] as well as recent experiments on granular superconductors,[7] indicate that for length scales below ξ (the percolation correlation length), and well above the microscopic lattice distance a, the clusters are self-similar fractals, as first proposed in Ref. 1. The same is true of the cluster backbones, obtained by erasing the dangling bonds. The backbone is responsible for the time-independent transport properties (e.g., of resistor networks) and for correlation functions (e.g., of magnetic moments) near the percolation threshold. The critical effects near percolation arise from length scales up to ξ. Bulk properties are obtained by juxtaposition of pieces of size ξ.

An important characteristic of a fractal is its order of ramification.[1] When a structure has a range of self-similarity between a microscopic length a and a finite scale ξ, the order of ramification at the point P is[1,2] the smallest number of interactions one must cut to isolate an (otherwise arbitrary) bounded subset that surrounds P and falls in the scaling range. When the smallest value of R over the structure, $R_{\rm min}$, satisfies $R_{\rm min} = 2$, the structure is *quasi one dimensional*. When the largest value of R over the structure, $R_{\rm max}$, satisfies $R_{\rm max} < \infty$, the structure is *finitely ramified*. MC simulations[6] and experimental results[7] suggest[1] that the backbone at p_c is finitely ramified[8] but not quasi one dimensional.[9]

The present Letter investigates certain nonrandom fractal lattices that we believe are the simplest compatible with the above properties. Specifically, (a) $R_{\rm min} = d+1$, which we take to be the lowest value for fully d-dimensional structures, and (b) $R_{\rm max} = 2R_{\rm min} - 2 = 2d$, which is the lowest value compatible with the above $R_{\rm min}$.[1] In fact, these are the only fractals we know to satisfy (a) and (b).

Thanks to its simplicity, we can test this family as a possible model for the backbone of the infinite cluster near the percolation threshold, up to the scale of ξ. It yields values for the exponents β' [describing the probability of belonging to the backbone, written as $B(p) \sim (p - p_c)^{\beta'} \sim \xi^{-\beta'/\nu}$] and t [the conductivity exponent defined through $\sigma(p) \sim (p - p_c)^t \sim \xi^{-t/\nu} = \xi^{-\tilde{t}}$]. These values agree with the MC results for the backbone at low d. As d increases, the MC results and the values of our model deviate from each other. Our model is designed to be made solely of interconnected loops, while actual backbones also include quasi one-dimensional links.[6,10] This structure, and the inclusion of randomness, may be necessary for a more detailed study of the backbone, especially at high dimensionalities. However, our results suggest that at low dimensionalities loops alone yield a reliable description of the backbone.

Observe that our model is the logical opposite of the simple "nodes and links" picture,[11,12] in which quasi one-dimensional links of length $L \sim (p-p_c)^{-\zeta}$ connect nodes of distance ξ apart. This simple model fails badly for low d,[13] where three different values of ζ are needed to explain all the observed properties.[14] Recently, Coniglio[10b] replaced the links by more complicated structures, and related each value of ζ to a different geometrical feature of these structures. However, even these sophisticated "node and link" pictures are characterized by $R_{\min}=2$, while we aim at structures with $R_{\min}>2$.[9]

As illustrated for $d=2$ in Fig. 1, our geometrical construction starts with a d-dimensional hypertetrahedron. The midpoints of the edges are connected, creating $d+1$ small hypertetrahedra. The volume at the center (bounded by faces of these new tetrahedra) is then erased, and the procedure is applied to the resulting $d+1$ new tetrahedra. This procedure is repeated down to the microscopic length scale, a. Each step changes the length scale by a factor 2, and creates $d+1$ new units. Therefore,[1] the fractal dimensionality[3] is given by $D=\ln(d+1)/\ln 2$.

Note that the order of ramification is sensitive to the detail of the construction. Here it is finite, equal to either $d+1$ or $2d$, but if the fraction of eliminated volume is slightly decreased (increased), R becomes zero (infinite).

Models with finite $R_{\max}$ can be solved exactly. We consider the conductivity of a dilute resistor network, $\sigma(p)$, the probability of a bond's belonging to the backbone, $B(p)$, and the crossover of dilute magnetic spin models from percolative behavior at zero temperature to thermal behavior at finite T, which occurs at $\tau \propto (p-p_c)^{\phi}$, with $\tau \sim k_B T/J$ for Heisenberg-like ($n \geq 2$) systems and $\tau \sim \exp(-2J/k_B T)$ for Ising-like systems (J and T are the exchange and the temperature). The number of backbone bonds in a volume ξ^d is $B(p)\xi^d$. Writing it as ξ^D, we find that the fractal dimensionality[3] is $D=d-\beta'/\nu$.

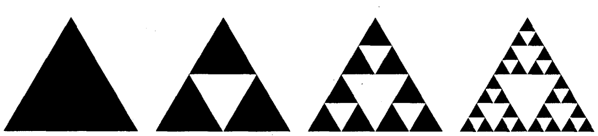

FIG. 1. The Sierpiński gasket's initial triangle, and the first construction stages. The limit shape's fractal dimensionality is $D=\ln 3/\ln 2 \cong 1.585$.

To evaluate the conductivity exponent $\tilde{t}$, the resistance of each bond before and after an iteration of our length rescaling transformation is denoted by $\rho(a)$ and by $\rho(ba)$. Here we shall use $b=2$. When a current I is sent into one corner of the tetrahedron, a current I/d flows out of each of the other d corners. With use of the symmetry of the problem, the voltages at the corners are easily calculated. The voltages at the corners remain unchanged after iteration, provided that $\rho(ba)=[(d+3)/(d+1)]\rho(a)$. Rewriting this as $\rho(ba)=b^{\tilde{\zeta}}\rho(a)$ yields, for $b=2$, $\tilde{\zeta}=\ln[(d+3)/(d+1)]/\ln 2$. Iterate l times, until $b^l=L/a$ and $\rho(L)\sim L^{\tilde{\zeta}}$. The conductivity of an equivalent homogeneous medium[11] is thus $\sigma(L)\sim L^{2-d}\rho(L)^{-1}$. Thus for length scales L below ξ, the conductivity behaves like $\sigma(L)\sim L^{-\tilde{t}}$, with $\tilde{t}=d-2+\tilde{\zeta}$. For scales L above ξ the conductivity ceases to change, and $\sigma \propto \xi^{-\tilde{t}}$. These relations (unlike the specific value of $\tilde{\zeta}$) are thus quite general, i.e., *independent of the particular geometrical model used.*[16] It would be nice to have numerical (or experimental) confirmation of the length scale dependence of $\sigma(L)$ for $L<\xi$.

In a dilute Heisenberg-like spin model, with exchange couplings J or zero (with probabilities p or $1-p$), the recursion relation for sufficiently small T is exactly the same for $(J/k_B T)$ as for $1/\rho$.[17] This yields the general result $\phi_H=\nu\tilde{\zeta}$.[10b] Available values of ϕ_H/ν are quoted below. [The recursion relation for $\exp(-2J/k_B T)$ in the Ising case is different, and its linear term is mainly connected to one-dimensional links.[10b] This explains why $1=\phi_I<\phi_H$.][18]

Discussion of Table I.—The values quoted for the backbone at $d=6$ are exact for percolation on a Cayley tree ($d=\infty$). For percolation on the *full* cluster, the upper critical dimensionality is $d=6$,[19] at which $\nu=\frac{1}{2}$ exactly. It has been conjectured[11] that the Cayley-tree value $t=3$ is also correct at $d=6$. We performed new MC simulations, and these confirm explicitly, for the first time, that the Cayley-tree result $\beta'=2$ applies at $d=6$. We took samples with lattice sizes of $L=6$, 8, 10, and 12, and used finite-size scaling to obtain $p_c=0.109$ and $\beta'/\nu=3.88^{+0.23}_{-0.34}$. The addition of logarithmic corrections increases this value slightly towards 4, yielding $\beta'=2$ and the random-walk dimensionality $D=2$. The same D is found in the simple "nodes and links" model, where $D=\zeta/\nu=1/\nu=2$.[11] The other values of β' in Table I come from earlier MC calculations.[6,20] The values for ν are estimates quoted in Refs. 5. The available series values[21] for t seem con-

TABLE I. Comparison between backbones and our models.

Backbone cluster						Our models	
d	β'	ν	t	$D=d-\beta'/\nu$	$\tilde{t}=t/\nu$	D	$\tilde{t}$
1 [a]	0	1	0	1	0	1	0
2	0.5–0.6	1.33–1.35	~1.1 [b]	1.5–1.6	~0.82	1.585	0.73
			~1.43 [c]		~1.06		
			0.9–1.4 [d]		0.67–1.05		
			1.5 [e]		1.1		
			1.33 [f]		1.0		
3	0.8–1.0	0.8–0.9	1.6–1.7 [b]	1.8–2.0	~1.9	2.000	1.58
			1.95 [c]		2.3		
			1.8 [e]		2.1		
4	1.0–1.2	~0.7	2.2 [b]	2.3–2.5	3.1	2.322	2.48
			2.37 [c]		3.4		
6	2	1/2	3	2	6	2.807	4.36

[a] Exact values.
[b] MC Refs. 6 and 22.
[c] Series, Ref. 21.
[d] From conductivity measurements, Refs. 23–25.
[e] From Heisenberg crossover measurement, Ref. 26, with use of $\varphi_H = \tilde{\zeta}\nu$.
[f] Ref. 27.

sistently higher than the MC values.[22] Experimental results are widely scattered, and seem to depend on details of the systems under study. These analyses did not include corrections to scaling,[21] and may also suffer from finite size and other difficulties. The discrepancies between the results of Heisenberg crossover and conductivity measurements, and between series and Monte Carlo data, will have to be resolved by better analyses.

The fractal dimensionalities and the orders of ramification of our models agree with those of the backbones for all $d \leq 4$. For $d \lesssim 3$ our values for $\tilde{t}$ agree surprisingly well with those given by MC and some of the experiments. The remaining disagreements between our values and those from the series (and other experiments), indicate (in our mind) that self-similar loops are the predominant feature of the backbone for $d \lesssim 3$, whereas nodes and quasi one-dimensional links must be added for higher d.

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[3] Self-similar fractals (Ref. 1) are often constructed by recursive replacement of segments, triangles, or squares by more complex sets made of smaller segments, triangles, or squares. When the number of subpieces is N and their linear relative scale is $1/b$, the fractal dimensionality D is defined by $b^D = N$.

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p. 79] found that the average number of distinct paths crossing a conducting sample, for both $d = 2$ and $d = 3$, is 1.2 ± 0.1.

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