[6]E. K. Warburton, J. W. Olness, and C. J. Lister, Phys. Rev. C 20, 619 (1979).
[7]S. J. Skorka, J. Hertel, and T. W. Retz-Schmidt, At. Data 2, 347 (1966).
[8]M. E. Rose, Phys. Rev. 76, 678 (1949).
[9]G. K. Horton and E. Phibbs, Phys. Rev. 96, 1066 (1954).
[10]E. L. Church and J. Weneser, Annu. Rev. Nucl. Sci. 10, 193 (1960).
[11]E. G. Adelberger, H. E. Swanson, M. D. Cooper, J. W. Tape, and T. A. Trainor, Phys. Rev. Lett. 34, 402 (1975), and Nuclear Physics Laboratory, University of Washington, Annual Report, 1978 (unpublished), p. 8.

Higher-Order Binding Corrections to the Lamb Shift

Jonathan Sapirstein
Department of Physics, University of California, Los Angeles, California 90024
(Received 4 September 1981)

A new method of calculating the one-loop self-energy contribution to the Lamb shift is presented. The technique relies on treating as perturbations certain terms in the equation for the Dirac Coulomb Green's function, in the absence of which the equation can be solved in terms of a simple integral representation. A new result for the 1*S* Lamb shift is obtained and compared with previous calculations.

PACS numbers: 31.30.Jv, 12.20.Ds

The classic high-precision tests of QED, the electron anomaly and the Lamb shift, have calculational difficulties of very different character.[1] The anomaly calculation involves the relevant external field, a constant magnetic field, only to first order, with energy shifts induced by repeated actions of the external field being totally negligible for laboratory magnetic fields. The difficulty of this calculation arises in treating the numerous complicated graphs associated with higher-order loops. On the other hand, the external field in the case of the Lamb shift, the Coulomb field of the nucleus, cannot be taken to act only once, or even a finite number of times, but must be taken into account to all orders even in the one-loop self-energy term. An attempt to expand in powers of the external field leads to a false expansion, in which the explicit powers of $Z\alpha$ from n Coulomb interactions, $(Z\alpha)^{2n+2}$, are compensated when the loop integration is performed by $(Z\alpha)^{-2(n-1)}$, making all terms $O((Z\alpha)^4)$.[2] This means that the full Green's function for the electron in an external Coulomb field must be used. Thus, even the lowest-order Lamb-shift calculation involves evaluating the Bethe logarithm, which is obtained as an integral over the nonrelativistic Coulomb Green's function. In the exact relativistic calculation, the Dirac Coulomb Green's function must be used, and unlike free-particle propagators, no closed form is known for this expression, although a form involving an infinite sum over partial waves does exist.[3] The lack of a convenient form for this Green's function is the essential difficulty of the Lamb-shift calculation. The main aim of this paper is to circumvent this difficulty by exploiting the fact that a simple integral representation can be given for the Dirac Coulomb Green's function if two terms, both of which are small if $Z\alpha \ll 1$, are treated as perturbations.

The usual approach to the evaluation of the 1*S* Lamb shift is to expand in powers of $Z\alpha$, the strength of the Coulomb potential, forming the series

$$\Delta E = \frac{m\alpha(Z\alpha)^4}{\pi}\{A_{40} + A_{41}\ln(Z\alpha)^{-2} + (Z\alpha)A_{50} + (Z\alpha)^2[A_{62}\ln^2(Z\alpha)^{-2} + A_{61}\ln(Z\alpha)^{-2} + G(Z\alpha)]\}, \tag{1a}$$

$$G(Z\alpha) = A_{60} + O(Z\alpha). \tag{1b}$$

The constant A_{40} includes the Bethe logarithm, and thus involves the full nonrelativistic Coulomb Green's function in its evaluation. However, of the remaining terms that have been directly evaluated, namely A_{41}, A_{50}, A_{62}, and A_{61}, the full Green's function is not needed, a fact that can be traced to the nonanalyticity of their coefficients in $(Z\alpha)^2$.[4] All such terms come from the action of three or fewer

Coulomb interactions. To directly evaluate A_{60}, however, another false expansion must be summed. There have been two approaches to this problem so far. Despite the complexity of the Dirac Coulomb Green's function, Mohr[5] was able to make a direct numerical evaluation of the one-loop self-energy graph. However, although there was no problem in evaluating the graph for $Z \geq 10$, numerical difficulties prohibited a direct evaluation for smaller Z's. By making an extrapolation of his results for $Z = 30$, 20, and 10, Mohr finds for the constant A_{60} [6]

$$A_{60} = -31.0 \pm 1.7. \tag{2}$$

This result is consistent with an earlier estimate of the constant in the $1S$ Lamb shift made by Erickson and Yennie,[7] but inconsistent with a subsequent refinement made by Erickson[8]:

$$A_{60} = -25.4 \pm 6.6, \tag{3a}$$

$$A_{60} = -25.79 \pm 0.67, \tag{3b}$$

respectively. While A_{60} is not directly evaluated in these works, large constants associated with the logarithmic terms are kept: The errors quoted are estimated bounds on the uncalculated terms.

The discrepancy between the results of Mohr and Erickson has until recently been academic, but recent experimental advances in determination of the Lamb shift[9] ($2S_{1/2}$-$2P_{1/2}$ splitting) require that the situation be resolved. The purpose of this paper is to make a direct evaluation of $A_{60}(1S)$.[10] My result is

$$A_{60} = -33.2 \pm 1.2, \tag{4}$$

which is in distinct disagreement with Eq. (3b) and, while somewhat smaller, is in basic agreement with Eq. (2). A small discrepancy with the result of Mohr may exist, but the large errors associated with numerical integrations do not allow a definite conclusion to be drawn on this point.

The Lamb-shift calculation involves the evaluation of

$$\Delta E_{1S} = -ie^2 \int \frac{d^4k}{(2\pi)^4} D_{\mu\nu}(k) \int d^3p\, d^3p'\, \bar{\psi}_{1S}(\vec{p}) \gamma_\mu S(\vec{p}-\vec{k}, \vec{p}'-\vec{k}; E-k_0) \gamma_\nu \psi_{1S}(\vec{p}'), \tag{5}$$

where $S(p, p'; E)$, the Dirac Coulomb Green's function, satisfies the equation

$$(E\gamma_0 - \vec{\gamma}\cdot\vec{p} - m) S(\vec{p}, \vec{p}'; E) + \frac{Z\alpha}{2\pi^2} \gamma_0 \int \frac{d^3q}{|\vec{p}-\vec{q}|^2} S(\vec{q}, \vec{p}'; E) = \delta^3(\vec{p}-\vec{p}'). \tag{6}$$

Now, it has been known for some time that the nonrelativistic Green's function equation,

$$(p_0^2 - p^2) G_0(\vec{p}, \vec{p}'; E) + \frac{\nu p_0}{\pi^2} \int \frac{d^3q}{|\vec{p}-\vec{q}|^2} G_0(\vec{q}, \vec{p}'; E) = 2m\delta^3(\vec{p}-\vec{p}'), \quad p_0 = (2mE)^{1/2}, \quad \nu = mZ\alpha/p_0, \tag{7}$$

can be solved in terms of an integral representation,[11]

$$G_0(\vec{p}, \vec{p}'; E) = \frac{\delta^3(\vec{p}-\vec{p}')}{p_0^2 - \vec{p}^2} - \frac{\nu p_0}{\pi^2} \frac{1}{|\vec{p}-\vec{p}'|^2} \frac{1}{p_0^2 - \vec{p}^2} \frac{1}{p_0^2 - \vec{p}'^2}$$
$$- \frac{4\nu^2 p_0^3}{\pi^2} \frac{1}{p_0^2 - \vec{p}^2} \frac{1}{p_0^2 - \vec{p}'^2} \int_0^1 d\rho\, \rho^{-\nu} \frac{1}{(1-\rho)^2 (p_0^2 - \vec{p}^2)(p_0^2 - \vec{p}'^2) - 4\rho p_0^2 |\vec{p}-\vec{p}'|^2}. \tag{8}$$

The invariance of the nonrelativistic Coulomb system under an O(4) symmetry is used in obtaining this relatively simple form. Since the deviations from the nonrelativistic energy spectrum arising from use of the Dirac equation are $O((Z\alpha)^2\ \mathrm{Ry})$, it should be possible for small-Z atoms to exploit their closeness to the nonrelativistic system. To do this, we define

$$S(\vec{p}, \vec{p}'; E) \equiv (E\gamma_0 - \vec{\gamma}\cdot\vec{p} + m) G(\vec{p}, \vec{p}'; E) + \frac{Z\alpha\gamma_0}{2\pi^2} \int \frac{d^3q}{|\vec{p}-\vec{q}|^2} G(\vec{q}, \vec{p}'; E) \tag{9}$$

and find that $G(\vec{p}, \vec{p}'; E)$ satisfies

$$[(E^2 - m^2) - \vec{p}^2] G(\vec{p}, \vec{p}'; E) + \frac{EZ\alpha}{\pi^2} \int \frac{d^3q}{|\vec{p}-\vec{q}|^2} G(\vec{q}, \vec{p}'; E)$$
$$+ \int d^3q \left[\frac{(Z\alpha)^2}{4\pi} \frac{1}{|\vec{p}-\vec{q}|} + \frac{Z\alpha\vec{\alpha}\cdot(\vec{p}-\vec{q})}{2\pi^2 |\vec{p}-\vec{q}|^2} \right] G(\vec{q}, \vec{p}'; E) = \delta^3(\vec{p}-\vec{p}'). \tag{10}$$

Aside from the presence of the last two terms in square brackets, to which we refer henceforth as M_1 and M_2, respectively, after scaling out a factor of $2m$ this equation is identical to (7), with the identifications

$$p_0^2 = E^2 - m^2, \quad \nu = EZ\alpha/p_0. \tag{11}$$

Therefore, following the approach of Hostler,[12] one can develop a perturbation expansion for $G(\vec{p}, \vec{p}'; E)$ in terms of M_1, M_2, and $G_0(\vec{p}, \vec{p}'; E)$. In order to calculate to $O(\alpha(Z\alpha)^6)$, five perturbations must be taken into account: M_1 and M_2 each once, M_1 together with M_2, and M_2 occurring two or three times. These perturbations lead to integrals similar to the one that gives the Bethe logarithm, with the difference that instead of a single Green's function being integrated over, a perturbation term is sandwiched between two such functions. All integrals $O(\alpha(Z\alpha)^6)$ are performed numerically,[13] while lower-order contributions are calculated analytically. The Coulomb gauge was found to be most convenient for this calculation. A major advantage of the form in Eq. (8) is that the terms where one Coulomb interaction, or none, occurs are naturally separated out, facilitating the treatment of mass renormalization and cancelation of spurious $(Z\alpha)^2$ terms. A detailed description of the calculation will be given elsewhere.

The result in Eq. (4) has been calculated only for the $1S$ state of hydrogen, and so no direct confrontation with experiments measuring the $1S_{1/2}$-$2P_{1/2}$ splitting is yet possible. However, the state dependence of A_{60} is expected to be small, and if one assumes that $A_{60}(1S_{1/2}) = A_{60}(2S_{1/2}) - A_{60}(2P_{1/2})$,[14] this result would give

$$\$ = 1057.860(9) \text{ MHz}, \tag{12}$$

which is in reasonable agreement with recent measurements.[9] However, there remain uncomputed corrections to the Lamb shift, most notably, in terms of numerical significance, the first-order binding correction to the two-loop self energy, which can be expected to give contributions on the order of 10 kHz. Therefore, while the present calculation makes it likely that an apparent discrepancy between theory and experiment is reduced, much theoretical work remains to be done before a decisive confrontation between theory and experiment at the level of a few kilohertz will be possible.

I gratefully acknowledge conversations with Stan Brodsky, Bill Caswell, Glen Erickson, Chris Fronsdal, Peter Lepage, Peter Mohr, Don Yennie, and in particular Greg Adkins, whose enthusiasm and help were vital to the completion of this work. I also acknowledge the generosity of Peter Schlein, who made available the Nova Eclipse S200 on which the bulk of the numerical computations were done. This work was supported in part by the National Science Foundation under Grant No. PHY 80-20144 A01.

[1]A recent review of QED is given by G. P. Lepage and D. R. Yennie, in Proceedings of the Second International Conference on Precision Measurement and Fundamental Constants, National Bureau of Standards, Gaithersburg, Maryland, June 1981 (to be published), Cornell Laboratory of Nuclear Studies Report No. 81/499.

[2]A lucid description of the power counting involved in the false expansion is given by H. M. Fried, and D. R. Yennie, Phys. Rev. 112, 1391 (1958).

[3]E. H. Wichmann and N. M. Kroll, Phys. Rev. 101, 843 (1956).

[4]A. J. Layzer, J. Math. Phys. 2, 292 (1961).

[5]P. J. Mohr, Ann. Phys. (N.Y.) 88, 52 (1974). References to earlier work on the high-Z behavior of the self-energy graph can be found in this paper.

[6]P. J. Mohr, private communication. The extrapolation method is as described in P. J. Mohr, Phys. Rev. Lett. 34, 1050 (1975).

[7]G. W. Erickson and D. R. Yennie, Ann. Phys. (N.Y.) 35, 271 (1965).

[8]G. W. Erickson, Phys. Rev. Lett. 27, 780 (1971).

[9]S. R. Lundeen and F. M. Pipkin, Phys. Rev. Lett. 46, 232 (1981). For further references, see Ref. 1.

[10]The calculation of $A_{60}(2S_{1/2}) - A_{60}(2P_{1/2})$ is in progress.

[11]J. Schwinger, J. Math. Phys. 5, 1606 (1964); L. Hostler, J. Math. Phys. 5, 1235 (1964).

[12]L. Hostler, J. Math. Phys. 5, 591 (1964).

[13]The numerical integration programs used were SHEP, described by A. Dufner, in Proceedings of the Colloquium on Computational Methods in Theoretical Physics, Marseille, 1970 (unpublished), and VEGAS [G. P. Lepage, J. Comp. Phys. 27, 192 (1978)].

[14]For example, in Ref. 6 Mohr finds $A_{60}(2S_{1/2}) - A_{60}(2P_{1/2}) = -31.2 \pm 1.6$, as compared to $A_{60}(1S) = -31.0 \pm 1.7$.