[8]J. Bernasconi, W. R. Schneider, and W. Wyss, Z. Phys. B 37, 175 (1980).
[9]S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, Rev. Mod. Phys. 53, 175 (1981).
[10]S. Alexander and J. Bernasconi, J. Phys. C 12, L1 (1979).
[11]J. Bernasconi, H. U. Beyeler, S. Strässler, and S. Alexander, Phys. Rev. Lett. 42, 819 (1979).
[12]G. Grüner, Bull. Am. Phys. Soc. 25, 255 (1980), and to be published.
[13]S. Alexander, J. Bernasconi, R. Biller, W. G. Clark, G. Grüner, R. Orbach, W. R. Schneider, and A. Zettl, Phys. Rev. B (to be published).
[14]Note that the proof of Eq. (5) is nontrivial and rather involved (W. R. Schneider, unpublished).
[15]It is possible to construct class (a) probability densities $\rho(w)$ such that the moments $\langle g_L{}^n\rangle$ do not exist for any finite L. They are, however, only of mathematical interest.
[16]C. Fox, Trans. Am. Math. Soc. 98, 395 (1961); K. C. Gupta and U. C. Jain, Proc. Nat. Inst. Sci. India A36, 594 (1966).

Approximate Effective Action for Quantum Gravity

Bryce S. DeWitt
Department of Physics, University of Texas, Austin, Texas 78712
(Received 5 October 1981)

A new gauge-invariant effective action is proposed for quantum gravity, based on older results that go beyond finite-order perturbation theory. Expressed in coordinate space rather than momentum space it should find important applications in theory of the early universe.

PACS numbers: 04.60.+n, 11.10.Np, 95.30.Sf, 98.80.Dr

Attempts to determine the dynamical behavior of the universe in its early moments have heretofore been based on the semiclassical approximation[1-11]: One computes particle production and vacuum polarization in a given time-varying background metric, constructs an expectation value $\langle T^{\mu\nu}\rangle$ for the stress tensor, and then adjusts the metric in such a way as to satisfy the self-consistency condition[12] $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G \times\langle T^{\mu\nu}\rangle$.

The semiclassical method is a one-loop approximation to the full theory.[13] It unfortunately suffers from ambiguity. Firstly, $\langle T^{\mu\nu}\rangle$ is not invariant under quantum field redefinitions that involve the background metric[14]; it can be made invariant only by including graviton loops. Secondly, $\langle T^{\mu\nu}\rangle$ must be regularized and renormalized. Renormalization involves the subtraction of terms that do not appear in the classical action and hence cannot be absorbed in parameter redefinitions. New arbitrary parameters make their appearance.

The need for graviton loops reminds us that we cannot study the early universe without quantizing the gravitational field itself. The appearance of arbitrary parameters reflects the nonrenormalizability of quantum gravity and tells us that we must go beyond one loop, indeed beyond finite-order perturbation theory.

The chief theoretical tool for studying quantum corrections to classical dynamical behavior is the *effective action.* A new theory of the effective action for gauge fields, including the gravitational field, has recently been worked out to all orders.[15,16] A set of computational rules exists leading to an effective action of the form $\Gamma = S+\Sigma$, where both the classical action S *and* the quantum correction Σ are gauge invariant. I do not discuss these rules here, beyond remarking that in Yang-Mills theory they greatly simplify the renormalization program, completely bypassing BRS techniques.[17,18] In this note I describe the general structure of Σ in gravity theory and suggest an approximate form for it based on nonperturbative results already obtained a number of years ago.

The construction of Σ requires the introduction of gauge-breaking terms and ghost propagators that are *covariant with respect to the effective metric* $g_{\mu\nu} \overset{\text{def}}{=} \langle \text{out}|g_{\mu\nu}|\text{in}\rangle/\langle\text{out}|\text{in}\rangle$.[19] The functional form of Σ is not independent of the choice of these terms. However, the solutions of the *effective field equation*

$$0 = \frac{\delta\Gamma}{\delta g_{\mu\nu}} = \frac{\delta S}{\delta g_{\mu\nu}} + \frac{\delta\Sigma}{\delta g_{\mu\nu}} \tag{1}$$

can be shown to be the same for all choices.[15,16,20] Furthermore, Σ is gauge invariant for all choices. In gravity theory this is expressed by the identity[21]

$$(\delta\Sigma/\delta g_{\mu\nu})_{;\nu} \equiv 0, \tag{2}$$

the covariant derivative being that defined by the effective metric.

We first study Σ near flat space-time with $\mathcal{R}^4$ topology. I *assume* that the Minkowski metric $\eta_{\mu\nu}$ is a stable solution of the effective field equation (1) just as it is of the classical field equation $\delta S/\delta g_{\mu\nu}=0$. That is, I assume that the Poincaré group, which is relevant for asymptotically flat space-times, is not dynamically broken. Then

$$(\delta\Sigma/\delta g_{\mu\nu})_{g_{\mu\nu}=\eta_{\mu\nu}} = 0, \tag{3}$$

whence, in virtue of (2),

$$\left[\left(\frac{\delta^2\Sigma}{\delta g_{\mu\nu}\delta g_{\sigma'\tau'}}\right)_{;\nu}\right]_{g_{\mu\nu}=\eta_{\mu\nu}}=0. \tag{4}$$

Denote by $\Sigma^{\mu\nu\sigma\tau}(p)$ the Fourier transform of $(\delta^2\Sigma/\delta g_{\mu\nu}\,\delta g_{\sigma'\tau'})_{g=\eta}$, with the δ function expressing momentum conservation removed. Equation (4) is equivalent to $\Sigma^{\mu\nu\sigma\tau}(p)p_\nu=0$, of which the general solution, respecting Lorentz invariance and the index symmetries of $\Sigma^{\mu\nu\sigma\tau}$, is

$$\begin{aligned}\Sigma^{\mu\nu\sigma\tau}(p) = {}&[(\eta^{\mu\sigma}\eta^{\nu\tau}+\eta^{\mu\tau}\eta^{\nu\sigma})p^4-(\eta^{\mu\sigma}p^\nu p^\tau+\eta^{\mu\tau}p^\nu p^\sigma+\eta^{\nu\sigma}p^\mu p^\tau+\eta^{\nu\tau}p^\mu p^\sigma)p^2+2p^\mu p^\nu p^\sigma p^\tau]\bar\Sigma_1(p^2)\\ &-[\eta^{\mu\nu}\eta^{\sigma\tau}p^4-(\eta^{\mu\nu}p^\sigma p^\tau+\eta^{\sigma\tau}p^\mu p^\nu)+p^\mu p^\nu p^\sigma p^\tau]\bar\Sigma_2(p^2),\end{aligned} \tag{5}$$

$\bar\Sigma_1$ and $\bar\Sigma_2$ being the two form factors of the graviton.

If the usual covariant gauge-breaking term is employed, yielding $(\eta_{\mu\sigma}\eta_{\nu\tau}+\eta_{\mu\tau}\eta_{\nu\sigma}-\eta_{\mu\nu}\eta_{\sigma\tau})/p^2$ for the conventionally scaled bare graviton propagator, then expression (5) leads to a full graviton propagator of the form

$$\begin{aligned}&[\eta_{\mu\sigma}\eta_{\nu\tau}+\eta_{\mu\tau}\eta_{\nu\sigma}-\tfrac{2}{3}\eta_{\mu\nu}\eta_{\sigma\tau}+\mu_P^{-2}(\eta_{\mu\sigma}p_\nu p_\tau+\eta_{\mu\tau}p_\nu p_\sigma+\eta_{\nu\sigma}p_\mu p_\tau+\eta_{\nu\tau}p_\mu p_\sigma)\Sigma_1(p^2)][p^2+\mu_P^{-2}p^4\Sigma_1(p^2)]^{-1}\\ &-\tfrac{1}{3}\eta_{\mu\nu}\eta_{\sigma\tau}[p^2+\mu_P^{-2}p^4\Sigma_2(p^2)]^{-1}-\tfrac{2}{3}\mu_P^{-2}[(\eta_{\mu\nu}p_\sigma p_\tau+\eta_{\sigma\tau}p_\mu p_\nu)p^2+2p_\mu p_\nu p_\sigma p_\tau][\Sigma_1(p^2)-\Sigma_2(p^2)]\\ &\times[p^2+\mu_P^{-2}p^4\Sigma_1(p^2)]^{-1}[p^2+\mu_P^{-2}p^4\Sigma_2(p^2)]^{-1},\end{aligned} \tag{6}$$

where

$$\Sigma_1=4\bar\Sigma_1,\quad \Sigma_2=3\bar\Sigma_2-2\bar\Sigma_1, \tag{7}$$

and where I have chosen the slightly unconventional definition

$$\mu_P=(16\pi G)^{-1/2} \tag{8}$$

for the Planck mass. In order that there be no physical ghosts in the theory neither $p^2+\mu_P^{-2}p^4\Sigma_1(p^2)$ nor $p^2+\mu_P^{-2}p^4\Sigma_2(p^2)$ may have any zeros on the real axis in the complex p^2 plane, other than $p^2=0$.

In the one-loop approximation of pure quantum gravity both Σ_1 and Σ_2, after renormalization, take the form const $\ln(p^2/\mu^2)$, where μ is an arbitrary scale parameter. Since the form factors become important only when $|p^2|$ is of the order of μ_P^2 and since experimental masses are negligible compared to μ_P, the factor $\ln(p^2/\mu^2)$ remains effectively unchanged if the one-loop contributions of all other fields are added to Σ_1 and Σ_2. Only the constant multiplying $\ln(p^2/\mu^2)$ changes. If one could believe the one-loop approximation then this constant would have to lie between 0 and $(\mu_P^2/\mu^2)e$ for physical ghosts to be absent. Alternatively, this could be regarded as a constraint on μ.

The one-loop approximation is of course not believable, among other reasons because μ *is* arbitrary. Since quantum gravity is not perturbatively renormalizable we must leave Σ_1 and Σ_2 unsubtracted and attempt to get finite answers from the theory by nonperturbative means. It is convenient to transform from momentum space back to ordinary space-time. It is not difficult to show that if Σ is expanded as a functional power series in $\varphi_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}$ then the term of lowest order is quadratic in $\varphi_{\mu\nu}$ and is uniquely determined by Eqs. (5) and (7) to have the form

$$\Sigma_{\rm quad}=\int d^4x\int d^4x'\,[\tfrac{1}{2}\tilde\Sigma_1((x-x')^2)C_{\mu\nu\sigma\tau}(x)C^{\mu\nu\sigma\tau}(x')-\tfrac{1}{6}\tilde\Sigma_2((x-x')^2)R(x)R(x')], \tag{9}$$

where $C_{\mu\nu\sigma\tau}$ is the linearized Weyl tensor, R is the linearized curvature scalar, and $\tilde\Sigma_1$ and $\tilde\Sigma_2$ are the Fourier transforms of Σ_1 and Σ_2, respectively.[22]

In the one-loop approximation *without* subtraction, the dominant singularities of both $\tilde\Sigma_1$ and $\tilde\Sigma_2$

are proportional to $i/(x-x')^4$. This singularity structure, which renders expression (9) logarithmically divergent, arises from products of pairs of Green's functions $i/(x-x')^2$, together with loop factors $-i$, in typical self-energy graphs. How does it get modified in the exact theory?

A partial answer to this question is known[23,24] in the case of ladder graphs in which the free ends at the top of each ladder are joined together to make a single line, leaving only the two free ends at the bottom. The dominant high-energy contribution to the infinite sum of *all* such graphs can be expressed as the solution of a single integral equation. The line at the top contributes a factor $i/(x-x')^2$ as always, but the rungs, when summed to all orders, contribute a factor $i/[(x-x')^2-\lambda_P{}^2]$, where

$$\lambda_P=(2\pi\mu_P)^{-1}=1.82\times10^{-33}\ \text{cm}. \tag{10}$$

The singularity of the rung factor lies on a hyperboloid at a distance λ_P outside the Minkowski light cone and implies *noncausal* propagation relative to Minkowski space-time. This is neither surprising nor alarming. When the metric itself undergoes quantum fluctuations "real" space-time is Minkowskian only in an averaged sense.

These results suggest that $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ may be well approximated by choosing each to be proportional to $i/(x-x')^2[(x-x')^2-\lambda_P{}^2]$ or, equivalently, to

$$\lambda_P{}^{-2}\left[\frac{i}{(x-x')^2-\lambda_P}-\frac{i}{(x-x')^2}\right]=i\int_0^1\frac{d\xi}{[(x-x')^2-\xi\lambda_P{}^2]^2}. \tag{11}$$

The integral on the right of (11) gives concrete expression to the old idea that quantum gravity "smears" the light cone.[25-28] A more complete theory, which sums other graphs besides ladder graphs, would presumably insert a smearing function $w(\xi)$ in the integrand.

Expressions (9) and (11) admit of immediate generalization to an approximation for Σ, and hence for Γ, that is invariant under the full diffeomorphism group (the gauge group of gravity theory):

$$\Gamma\approx\mu_P{}^2\int g^{1/2}R\,d^4x+\mu_P{}^2\int d^4x\int d^4x'\,g^{1/2}(x)g^{1/2}(x')\left[\frac{i}{\sigma(x,x')-\frac12\lambda_P{}^2+i0}-\frac{i}{\sigma(x,x')+i0}\right]$$
$$\times\left[\tfrac14A_1g^{\mu\alpha'}g^{\nu\beta'}g^{\sigma\gamma'}g^{\tau\delta'}C_{\mu\nu\sigma\tau}(x)C_{\alpha'\beta'\gamma'\delta'}(x')-\tfrac{1}{12}A_2R(x)R(x')\right]. \tag{12}$$

Here g is $-\det(g_{\mu\nu})$, $g^{\mu\alpha'}$ is the parallel displacement bivector,[29] $\sigma(x,x')$ is half the square of the geodetic distance between x and x',[29] and $C_{\mu\nu\sigma\tau}$ and R are the Weyl tensor and curvature scalar of the full nonlinear theory. A_1 and A_2 are numerical coefficients whose precise values depend on the numbers and kinds of matter fields included, but whose magnitudes are not vastly different from unity. The $i0$ in the "propagators" specifies how the poles are to be skirted in the double integral, and the other factors i remind us that both Γ and the effective field, which is an "in-out" average, are generally complex valued.

Although expression (12) has been derived by arguments starting from flat space-time, I propose that it be taken seriously even under conditions of strong curvature ($R_{\mu\nu\sigma\tau}\gtrsim\mu_P{}^2$) and with topologies other than $\mathbb{R}^4$. Efforts are currently under way to test it on compact Robertson-Walker universes to see whether, under generic realistic conditions, it will suppress the initial curvature singularity. Among the properties of Robertson-Walker models that simplify this investigation is conformal flatness. The Weyl tensor disappears from expression (12), taking with it the parallel displacement bivectors, leaving $\sigma(x,x')$ as the only difficult geometrical quantity to compute and A_2 as the only adjustable constant.

Expression (12), based as it is on a quadratic approximation to Σ that is determined solely by the graviton propagator, cannot be expected to yield accurate vertex functions (third functional derivatives and higher). Nevertheless it is well known[30] that in regions of momentum space where $\Sigma_1(p^2)$ and $\Sigma_2(p^2)$ are slowly varying, e.g., in the ultrahigh-energy region $|p^2|\gg\mu_P{}^2$ (see below), the vertex functions *are* fully determined by the graviton propagator in virtue of the gauge-invariance condition (2). Therefore, if $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are well approximated by (11) then expression (12) *has* the correct structure as $x'\to x$ and will yield qualitatively correct dynamical behavior. More accurate vertex functions at lower energies could in principle be obtained by adding to expression (12) higher-multiple integrals in which the curvature appears cubically, quartically, etc., along with factors involving $g^{\mu\alpha'}$ and $\sigma(x,x')$.

It is not difficult to verify that expression (12) yields the graviton form factors $\Sigma_1(p^2)=A_1F(p^2)$, $\Sigma_2(p^2)=A_2F(p^2)$, where

$$F(p^2)=-\frac{\pi i}{2}\frac{H_1^{(2)}((\lambda_P^2p^2-i0)^{1/2})}{(\lambda_P^2p^2-i0)^{1/2}}-\frac{1}{\lambda_P^2p^2-i0}. \tag{13}$$

The function $F(p^2)$ is complex for spacelike momenta (and hence yields no tachyon ghosts) and real for timelike momenta. In both cases it damps to zero as $|p^2|\to\infty$. It yields no timelike ghosts provided A_1 and A_2 lie between $-\infty$ and $1/(2\pi)^2$. This may be regarded as a constraint on the number and kinds of fields that nature will permit in addition to the gravitational field.

This work was supported in part by grants from the National Science Foundation.

[1]Ya. B. Zel'dovich and A. A. Starobinsky, Zh. Eksp. Teor. Fiz. 61, 2161 (1972) [Sov. Phys. JETP 34, 1159 (1972)].

[2]L. Parker and S. A. Fulling, Phys. Rev. D 7, 2357 (1973).

[3]J. B. Hartle, Phys. Rev. Lett. 39, 1373 (1977).

[4]B. L. Hu and L. Parker, Phys. Rev. B 17, 933 (1978).

[5]M. V. Fischetti, J. B. Hartle, and B. L. Hu, Phys. Rev. D 20, 1757 (1979).

[6]J. B. Hartle and B. L. Hu, Phys. Rev. D 20, 1772 (1979).

[7]J. B. Hartle and B. L. Hu, Phys. Rev. D 21, 2756 (1980).

[8]J. B. Hartle, Phys. Rev. D 22, 2091 (1980).

[9]J. B. Hartle, Phys. Rev. D 23, 2121 (1981).

[10]A. A. Starobinsky, Phys. Lett. 91B, 99 (1980).

[11]B. L. Hu, Phys. Lett. 103B, 331 (1981).

[12]I adopt the metric signature $-+++$ and units for which $\hbar=c=1$. Conventions for the Reiemann, Weyl, and Ricci Tensors, and for the curvature scalar, are those of C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[13]B. S. DeWitt, in *General Relativity, An Einstein Centenary Survey*, edited by S. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, 1979).

[14]M. Duff, in *Quantum Gravity II*, edited by C. Isham, R. Penrose, and D. Sciama (Oxford Univ. Press, New York, 1981).

[15]G. 't Hooft, in *Functional and Probabilistic Methods in Quantum Field Theory*, edited by B. Jancewicz, Proceedings of the Twelfth Winter School of Theoretical Physics, Karpacz, 1975 (Wydawnictwa Uniwersytetu Wrocławskiego, Wrocław, 1976), Vol. I.

[16]B. S. DeWitt, in *Quantum Gravity II*, edited by C. Isham, R. Penrose, and D. Sciama (Oxford Univ. Press, New York, 1981).

[17]L. F. Abott, Nucl. Phys. B185, 189 (1981).

[18]C. Hart, Ph.D. thesis, University of Texas, 1981 (unpublished). For BRS techniques see B. W. Lee, in *Methods in Field Theory*, edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976).

[19]The effective metric here replaces the classical background metric of the one-loop approximation. Since the effective metric itself depends on the choice of gauge breaking terms, a self-consistency condition is implicit.

[20]The S matrix (built out of the tree amplitudes of Γ) is also choice independent.

[21]Equation (2) holds strictly only in pure quantum gravity. In general, additional terms should appear, involving the matter fields. However, these additional terms vanish when the matter fields vanish, and none of the conclusions that follow are altered.

[22]The Ricci tensor $R_{\mu\nu}$ is absent from expression (9). Purely on the basis of gauge invariance one might suppose that a third form factor should be included, multiplying $R_{\mu\nu}(x)R^{\mu\nu}(x')$. There really *are* only two independent form factors, however. This is because the double integral $\int d^4x\int d^4x' f(x-x')[R_{\mu\nu\sigma\tau}(x)R^{\mu\nu\sigma\tau}(x') - 4R_{\mu\nu}(x)R^{\mu\nu}(x')+R(x)R(x')]$ *vanishes* for *all* functions $f(x-x')$. This integral is a generalization, in the linearized theory, of the well-known Gauss-Bonnet-Chern invariant for four manifolds.

[23]B. S. DeWitt, Phys. Rev. Lett. 13, 114 (1964).

[24]I. B. Khriplovich, Yad. Fiz. 2, 950 (1965) [Sov. J. Nucl. Phys. 3, 415 (1966)].

[25]W. Pauli, Helv. Phys. Acta Suppl. 4, 69 (1956).

[26]O. Klein, in *Niels Bohr and the Development of Physics*, edited by W. Pauli, L. Rosenfeld, and V. Weisskopf (Pergamon, New York, 1955), and Nuovo Cimento Suppl. 6, 344 (1957).

[27]L. D. Landau, in *Niels Bohr and the Development of Physics*, edited by W. Pauli, L. Rosenfeld, and V. Weisskopf (Pergamon, New York, 1955).

[28]S. Deser, Rev. Mod. Phys. 29, 417 (1957).

[29]B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965), Chap. 17.

[30]B. S. DeWitt, Phys. Rev. 162, 1239 (1967).