integer greater than 1. Then the resulting rotation number is

$$w(\hat{a},M) \approx w^*\{1+(-1)^M(w^*)^{2M+1}(\hat{a}-1) \times (1+2w^*)/[2+(\hat{a}-2)w^*]\}.$$

The corresponding critical value of k can be written as $k_c(\hat{a},M)=k_c(\infty)-\epsilon$, where $k_c(\infty)$ is the critical coupling when $w=w^*$.

According to the numerical work of Shenker and myself,[5] as one progresses through cycles of length q_n for $n \leq M-1$, the residues of the resulting cycles are, for small ϵ, a function of ϵq_n. This suggests that ϵ may be viewed as a scaling field[9] which grows as $\epsilon_n \sim \epsilon q_n \sim \epsilon (w^*)^{-n}$. Then at $n=M$ the kind of recursion changes since at that step $T_{M+1}=(T_M)^{\hat{a}} \cdot T_{M-1}$. This then produces a sudden and large change in the value of the scaling field in this and the next few steps in iteration. If $\hat{a}$ is of order unity, the scaling field will change by an amount which also is of order unity. But then as one goes on many more steps in iteration T_n must settle down to its original fixed point behavior, hence showing a scaling field $\epsilon_n = 0$ for $n \gg M$. If the change near step M is of order unity, the value $\epsilon_{M-1} \sim \epsilon (w^*)^{-M}$ must also be a number of order unity which depends on $\hat{a}$. Hence, we say $\epsilon_{M-1}=\alpha\epsilon(\hat{a})$ or

$$k_c(\hat{a},M)=k_c(\infty)-(w^*)^{-M}\alpha\,\delta\epsilon(\hat{a}) \tag{11}$$

for $M \gg 1$. Here $\delta\epsilon(\hat{a})$ is universal, but α is not.

This work was based upon a long and fruitful collaborative effort with Scott J. Shenker, which is in part reflected in our joint paper,[5] but which also includes many unpublished ideas which led up to the present paper. I have also had useful discussions with S. Aubry, J. Greene, M. Feigenbaum, and M. Widom.

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Classical Hopping Conduction in Random One-Dimensional Systems: Nonuniversal Limit Theorems and Quasilocalization Effects

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The $L\to\infty$ asymptotic properties of $\rho_L(g)$, the probability distribution of the classical hopping conductivity g_L corresponding to random one-dimensional systems of length L, are determined. These properties are nonuniversal, and become anomalous if the probability density $\rho(w)$ of the random near-neighbor hopping rates is such that $\int_0^\infty dw\,\rho(w)\,w^{-1}$ does not exist. The associated quasilocalization effects are discussed and their experimental observability is speculated upon.

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The transport properties of random one-dimensional systems represent a topic of high current interest, and the theoretical as well as the experimental situation is still quite controversial, in particular with respect to localization effects. For quantum systems, the zero-temperature resistance is expected[1,2] to increase exponentially with L, the length of the one-dimensional system, reflecting exponential localization of the electronic eigenstates. Analytical[2-4] and numerical[5] investigations, however, indicate that calculated quantities such as resistance or conductance do not obey a conventional law of large numbers, i.e., their distributions become singular as $L \to \infty$. In addition, the experimental situation with respect to the observation of localization effects in thin wires[6] is also rather confusing.

Analogous problems in classical disordered one-dimensional systems have not yet been investigated, and it is the purpose of this paper to present exact results for the asymptotic length dependence of the dc conductivity in a simple model for classical hopping-type transport.[7-9] The model is based on a master equation description of hopping transport on a one-dimensional lattice,

$$dP_n/dt = W_{n,n-1}(P_{n-1} - P_n) + W_{n,n+1}(P_{n+1} - P_n), \quad (1)$$

where n labels the lattice sites, and $P_n(t)$ is the probability of finding a hopping particle at site n at time t. The near-neighbor hopping rates, $W_{n,n+1} = W_{n+1,n} \geq 0$, are assumed to be mutually independent random variables, distributed according to a probability density $\rho(w)$. In the following, we shall show that this randomness can lead to localization or quasilocalization effects which are similar to (but qualitatively different from) those observed in quantum mechanical one-dimensional systems. Specifically, we shall determine and analyze, for several representative classes of $\rho(w)$, the exact $L \to \infty$ asymptotic form of the probability distribution ρ_L of the (normalized) dc hopping conductivity g_L corresponding to a system of finite length L.

Previous work on the above model[7-13] has mainly concentrated on time- and frequency-dependent properties of an *infinite system* described by Eqs. (1), e.g., on the behavior of the frequency-dependent hopping conductivity $\sigma(\omega)$. Via the fluctuation-dissipation theorem, this can be expressed as[8,9]

$$\sigma(\omega) = (n_0 e^2/k_B T) W_0 \langle g(\omega)\rangle, \quad (2)$$

where

$$g(\omega) = -\tfrac{1}{2}\omega^2 \sum_{n=-\infty}^{\infty} n^2 \tilde{P}_n(-i\omega), \quad (3)$$

and where the $\tilde{P}_n$ represent the solution of the Laplace transform of Eqs. (1), supplemented by the initial condition $P_n(0) = \delta_{n,0}$. In Eq. (2), n_0 denotes the density and e the charge of the hopping particles, T is the temperature, and the lattice constant is unity. W_0 denotes some representative hopping rate, so that $g(\omega)$ is a *normalized conductivity*. In the following, the $W_{n,n+1}$ thus represent normalized hopping rates (with respect to W_0), and the average $\langle \ldots \rangle$ is defined with respect to the distribution of these (independent) random variables. For three general classes of probability densities $\rho(w)$ (see below), the low-frequency behavior of $\langle g(\omega)\rangle$ has been determined[8,9] via a general scaling hypothesis from the exact asymptotic results for $\langle \tilde{P}_0(-i\omega)\rangle$. The corresponding expressions[8,9] imply, in particular, that the dc conductivity $\langle g(0)\rangle$ is given by $\langle g(0)\rangle = W_{av} > 0$ if $\rho(w)$ is such that $W_{av}^{-1} \equiv \int_0^\infty dw \times \rho(w) w^{-1} < \infty$, and that $\langle g(0)\rangle = 0$ if W_{av}^{-1} does not exist. In the latter case, the $\omega \to 0$ asymptotic behavior of $\langle g(\omega)\rangle$ is anomalous and nonuniversal, i.e., it depends explicitly on the analytic behavior of $\rho(w)$ near $w = 0$.

Under the assumption of a specific (temperature-dependent) hopping-rate distribution, this simple classical model has been shown[11] to lead to a remarkably accurate and detailed description of the peculiar frequency and temperature dependence of the complex electrical conductivity $\sigma(\omega, T)$ in the one-dimensional superionic conductor hollandite. Recently, the rather complicated behavior of $\sigma(\omega, T)$ in the quasi-one-dimensional electronic conductor quinolinium ditetracyanoquinodimethanide $[\mathrm{Qn(TCNQ)_2}]$ has also been analyzed in terms of such a classical hopping model.[12,13]

We now concentrate on *finite systems*, and on the dc conductivity,

$$g_L \equiv \lim_{\omega \to 0} g_L(\omega), \quad (4)$$

where L denotes the length of the system. To be definite, we define $g_L(\omega)$ as the normalized conductivity, given by Eq. (3), of an infinite system that consists of periodical repetitions of any array of L hopping rates $(W_1, W_2, \ldots, W_L)$. It follows[14] that g_L then simply becomes

$$g_L = L \cdot G_L = L \cdot \Big(\sum_{n=1}^{L} W_n^{-1}\Big)^{-1}, \quad (5)$$

i.e., G_L can be regarded as the "equivalent conductance" of L "conductances," $W_1, W_2, \ldots, W_L$, in series. For a given probability density ρ for the hopping rates, let ρ_L denote the probability density of the conductivity g_L, and R_L that of the conductance G_L. Equation (5) implies that $G_L = (G_{L-1}^{-1} + W_L^{-1})^{-1}$, so that R_L satisfies the recursive integral equation

$$R_L(x) = \int_0^\infty dx\, R_{L-1}(y) \int_0^\infty dz\, \rho(z)\delta(x - (y^{-1} + z^{-1})^{-1}), \quad (6)$$

and our aim is to determine the behavior of $\rho_L(x) = L^{-1}R_L(L^{-1}x)$, and of corresponding averages, at least asymptotically as $L \to \infty$. In particular, we shall explicitly discuss the average conductivity, $\langle g_L \rangle = \int_0^\infty dx\, \rho_L(x)x$, and the standard deviation, $\Delta g_L = (\langle g_L{}^2 \rangle - \langle g_L \rangle^2)^{1/2}$.

We first consider an *ordered system*, i.e., $\rho(w) = \delta(w - W)$. Then our problem becomes trivial, and one has $g_L = W$, independent of L. *Percolation systems*, for which $\delta(w) = p\delta(w) + (1-p)\delta(w - W)$, are also easily investigated, and for any finite L one finds

$$\rho_L(x) = (1 - e^{-\lambda L})\delta(x) + e^{-\lambda L}\delta(x - W), \qquad (7)$$

where $\lambda \equiv -\ln(1-p)$. It follows that $\langle g_L \rangle$ decreases exponentially with increasing L, $\langle g_L \rangle = W\exp(-\lambda L)$, but also that the standard deviation decreases slower than the mean, $\Delta g_L / \langle g_L \rangle \approx \exp(\frac{1}{2}\lambda L)$ as $L \to \infty$. Although the reason for the divergence of $\Delta g_L / \langle g_L \rangle$ is rather trivial in the present case, we may note that qualitatively similar observations are made in random quantum systems.[2-5]

We now turn to more interesting hopping-rate distributions, and concentrate on the three general classes of probability densities $\rho(w)$ that have been considered[7-9] in the investigations of the frequency-dependent properties of the model: *Class (a)* consists of all $\rho(w)$ for which

$$W_{\text{av}}{}^{-1} \equiv \int_0^\infty dw\, \rho(w) w^{-1} < \infty, \qquad (8)$$

i.e., $\rho(w)$ either has a lower cutoff, or at least goes to zero as $w \to 0$. *Class (b)* contains the $\rho(w)$ which remain finite as $w \to 0$,

$$\rho(w) \to \rho_0 = \text{const} > 0, \quad \text{as } w \to 0. \qquad (9)$$

Class (c) probability densities,

$$\rho(w) = (1-\alpha)w^{-\alpha}\Theta(1-w), \quad 0 < \alpha < 1, \qquad (10)$$

finally, exhibit a power law singularity as $w \to 0$. (Θ denotes the Heaviside step function.)

For these three classes of hopping-rate distributions it turns out that, asymptotically as $L \to \infty$, $\rho_L(x)$ approaches a homogeneous function representation,

$$\rho_L(x) = \lambda_L h(\lambda_L x), \qquad (11)$$

where the scaling factor λ_L as well as the limiting scaling function $h(x)$ are functionals of $\rho(w)$. They can be determined rigorously, e.g., by inserting an *Ansatz* of the form $R_L(x) = L\lambda_L f_L(L \times \lambda_L x)$ into the recursive integral equation for R_L, Eq. (6). The limiting integral equation for $h(x) = \lim_{L\to\infty} f_L(x)$ can then be solved by use of methods similar to those described in the appendix of Ref. 8. It is also possible to determine $h(x)$ and λ_L via a direct (nonrecursive) evaluation of the Laplace transform of R_L in the limit as $L \to \infty$. We finally note that for class (c) distributions the form of Eq. (11), with the correct scaling factor $\lambda_L = L^{\alpha/(1-\alpha)}$, is suggested by scaling arguments similar to those used in Refs. 9 and 10. The rather involved mathematical details of a rigorous determination of λ_L and $h(x)$ will be presented in a forthcoming publication, and in the following we restrict ourselves to a summary of the main explicit results.

For *class (a)* systems one obtains $h(x) = \delta(x-1)$ and $\lambda_L = W_{\text{av}}{}^{-1}$, where $W_{\text{av}}{}^{-1}$ [defined in Eq. (8)] is independent of L. If $\langle g_L \rangle$ and $\langle g_L{}^2 \rangle$ exist,[15] it follows that

$$\lim_{L\to\infty} \langle g_L \rangle = W_{\text{av}}, \quad \lim_{L\to\infty} \Delta g_L = 0. \qquad (12)$$

Asymptotically for large L, class (a) systems therefore behave as an ordered system with an average hopping rate W_{av}.

In *class (b)* systems one still has $h(x) = \delta(x-1)$, but now $\lambda_L = \rho_0 \ln L$. This implies a logarithmic decay of the average dc conductivity,

$$\langle g_L \rangle \approx (\rho_0 \ln L)^{-1}, \quad L \to \infty. \qquad (13)$$

Class (c) systems exhibit an asymptotic behavior which is intermediate between that of an ordered system and that of percolation systems. We obtain $\lambda_L = L^{\alpha/(1-\alpha)}$ and

$$h_\alpha(x) = \gamma\Gamma(\alpha)^\gamma H_{11}{}^{10}\left(\Gamma(\alpha)^\gamma x \middle| {(-1,1) \atop (-\gamma,\gamma)}\right),$$
$$\gamma = 1/(1-\alpha), \qquad (14)$$

where $H_{pq}{}^{mn}$ denotes the so-called H function of Fox,[16] a generalized hypergeometric function. It follows that $\langle g_L \rangle$ decreases according to a power law,

$$\langle g_L \rangle \approx \gamma\Gamma(\gamma)\Gamma(\alpha)^{-\gamma}L^{-\alpha/(1-\alpha)}, \quad L \to \infty, \qquad (15)$$

and that the standard deviation Δg_L becomes proportional to $\langle g_L \rangle$,

$$\lim_{L\to\infty} \Delta g_L / \langle g_L \rangle = [\gamma^{-1}\Gamma(2\gamma)\Gamma(\gamma)^{-2} - 1]^{1/2}. \qquad (16)$$

Equation (16) reflects the fact that for class (c) distributions the limiting scaling function $h(x)$ is not a delta function. For $\alpha = \frac{1}{2}$, e.g., we have $h_{1/2} = \frac{1}{2}x^{-1/2}\exp(-\frac{1}{4}\pi x)$, and $\langle g_L \rangle \approx (2/\pi)L^{-1}$ as $L \to \infty$.

If the inverse first moment, $W_{\text{av}}{}^{-1}$, of the hopping-rate distribution $\rho(w)$ does not exist, the average dc hopping conductivity $\langle g_L \rangle$ thus always decays to zero with increasing sample length L,

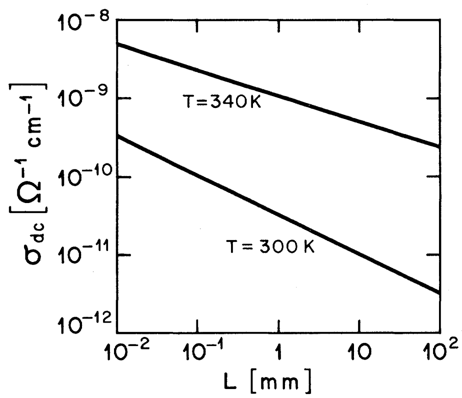

FIG. 1. Predicted dependence of the ionic dc conductivity σ_{dc} on sample length L for the one-dimensional superionic conductor hollandite. The underlying model assumptions are discussed in the text.

i.e., we observe interesting *quasilocalization effects* in a classical one-dimensional system. The asymptotic behavior of the conductivity distribution ρ_L is, however, nonuniversal and qualitatively different for different classes of $\rho(w)$. For class (b) distributions $\langle g_L\rangle$ decays logarithmically and $\lim_{L\to\infty}\Delta g_L/\langle g_L\rangle = 0$, whereas in class (c) systems $\langle g_L\rangle$ exhibits a power law decay and $\Delta g_L/\langle g_L\rangle$ approaches a finite constant. These results can be contrasted with the strong localization effects in percolation systems, where $\langle g_L\rangle$ decreases exponentially and $\Delta g_L/\langle g_L\rangle$ increases with increasing L.

While most distributions $\rho(w)$ of interest for physical problems either belong to one of the above classes or are of percolation type, it is interesting to point out that there exist $\rho(w)$ which lead to even stronger localization effects than the exponential ones obtained for percolation systems. For example, if $\rho(w)$ diverges as $w^{-1}|\ln w|^{-\nu}$ for $w\to 0$ ($\nu>1$), scaling arguments indicate that $\langle g_L\rangle$ should decay as $\langle g_L\rangle \sim L^{\nu/(1-\nu)} \times\exp(-L^{1/(\nu-1)})$, but the asymptotic properties of ρ_L seem to be very peculiar.

In conclusion, we have derived exact limit theorems for the conductivity distribution ρ_L in classical one-dimensional model systems. In contrast to quantum localization, which occurs for any type of disorder, we observe classical localization (or quasilocalization) effects only if the near-neighbor hopping-rate distribution is not of type class (a) [defined by Eq. (8) above]. It is therefore interesting to speculate whether such effects could actually be observed in real physical systems. The best candidates are one-dimensional superionic conductors such as hollandite, whose anomalous transport properties have successfully been analyzed[11] in terms of our classical hopping model. The specific model used in these investigations (thermally activated hopping over random barriers, and an exponential distribution of the barrier heights)[11] leads to a class (c) probability density $\rho(w)$, with a temperature-dependent exponent α, $\alpha = 1 - T/T_m$. Our results [see Eq. (15)] would thus imply that with increasing sample length L, the dc ionic conductivity should decrease as

$$\sigma_{dc}(T, L) = \sigma_0(T)L^{-(T_m-T)/T}, \quad T<T_m. \tag{17}$$

In Fig. 1 we display the predicted length dependence of σ_{dc} at $T = 300$ K and $T = 340$ K for the case of hollandite, where $T_m \approx 450$ K.[11] To evaluate $\sigma_0(T)$, the remaining parameter values have also been adopted from Ref. 11, where this idealized model has been shown to lead to a remarkably accurate description of the experimental results for $\sigma(\omega, T)$ over a wide frequency and temperature range. We remark, however, that these ac investigations[11] do not rule out, e.g., the possibility of a lower cutoff at some W_{min} in the hopping-rate distribution $\rho(w)$. The associated (L-independent) value for σ_{dc} would then prevent an observation of its L dependence above some critical length L_c. From Fig. 1 it follows that at $T = 300$ K we can only expect to observe an L dependence in samples of reasonable length ($L \gtrsim 1$ mm) if the W_{min}-induced σ_{dc} threshold is below about 10^{-11} (Ω cm)$^{-1}$. From the present experimental information,[11] however, we can only deduce an upper bound of about 10^{-9} (Ω cm^{-1}) for σ_{dc} at $T = 300$ K.

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Approximate Effective Action for Quantum Gravity

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A new gauge-invariant effective action is proposed for quantum gravity, based on older results that go beyond finite-order perturbation theory. Expressed in coordinate space rather than momentum space it should find important applications in theory of the early universe.

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Attempts to determine the dynamical behavior of the universe in its early moments have heretofore been based on the semiclassical approximation[1-11]: One computes particle production and vacuum polarization in a given time-varying background metric, constructs an expectation value $\langle T^{\mu\nu}\rangle$ for the stress tensor, and then adjusts the metric in such a way as to satisfy the self-consistency condition[12] $R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R=8\pi G\times\langle T^{\mu\nu}\rangle$.

The semiclassical method is a one-loop approximation to the full theory.[13] It unfortunately suffers from ambiguity. Firstly, $\langle T^{\mu\nu}\rangle$ is not invariant under quantum field redefinitions that involve the background metric[14]; it can be made invariant only by including graviton loops. Secondly, $\langle T^{\mu\nu}\rangle$ must be regularized and renormalized. Renormalization involves the subtraction of terms that do not appear in the classical action and hence cannot be absorbed in parameter redefinitions. New arbitrary parameters make their appearance.

The need for graviton loops reminds us that we cannot study the early universe without quantizing the gravitational field itself. The appearance of arbitrary parameters reflects the nonrenormalizability of quantum gravity and tells us that we must go beyond one loop, indeed beyond finite-order perturbation theory.

The chief theoretical tool for studying quantum corrections to classical dynamical behavior is the *effective action.* A new theory of the effective action for gauge fields, including the gravitational field, has recently been worked out to all orders.[15,16] A set of computational rules exists leading to an effective action of the form $\Gamma=S+\Sigma$, where both the classical action S *and* the quantum correction Σ are gauge invariant. I do not discuss these rules here, beyond remarking that in Yang-Mills theory they greatly simplify the renormalization program, completely bypassing BRS techniques.[17,18] In this note I describe the general structure of Σ in gravity theory and suggest an approximate form for it based on nonperturbative results already obtained a number of years ago.

The construction of Σ requires the introduction of gauge-breaking terms and ghost propagators that are *covariant with respect to the effective metric* $g_{\mu\nu}\overset{\mathrm{def}}{=}\langle\mathrm{out}|g_{\mu\nu}|\mathrm{in}\rangle/\langle\mathrm{out}|\mathrm{in}\rangle$.[19] The functional form of Σ is not independent of the choice of these terms. However, the solutions of the *effective field equation*

$$0=\frac{\delta\Gamma}{\delta g_{\mu\nu}}=\frac{\delta S}{\delta g_{\mu\nu}}+\frac{\delta\Sigma}{\delta g_{\mu\nu}} \tag{1}$$