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Scaling for a Critical Kolmogorov-Arnold-Moser Trajectory

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In problems involving two-dimensional area-preserving maps, stochastic regions are separated by continuous curves called Kolmogorov-Arnold-Moser trajectories. As the mapping is changed continuously, these regions may fuse via the disappearance of the intervening trajectories. Scaling arguments are presented to describe the behavior of the curves near their disappearance.

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The behavior of a dynamical system may conveniently be studied by replacing the continuous time evaluation by a discrete mapping in which the coordinates at a point in the trajectory are expressed as a function of the coordinates at an earlier point. For a Hamiltonian system with two degrees of freedom, the corresponding map expresses new coordinates r', θ' as a function of old coordinates $(r', \theta') = T(r, \theta)$. The Hamiltonian, and hence reversible, nature of the problem is expressed by the condition that the Jacobian $\partial(r', \theta')/\partial(r, \theta)$ equals 1—so that areas in the (r, θ) phase space are preserved.

The qualitative nature of flows in the Hamiltonian problem is reflected by the qualitative behavior of the corresponding mapping problem. In both problems, Kolmogorov-Arnold-Moser (KAM) trajectories[1,2] serve as an important classifying feature since no flow can cross the surface represented by this kind of trajectory. In the mapping context, a KAM trajectory is a continuous curve in the r, θ plane, say Γ, such that all points on Γ map into other points on Γ and are each the result of the application of the map T to some other point on Γ. Such a KAM trajectory can divide the r, θ plane into noncommunicating pieces, i.e., into discrete areas from which a point cannot escape under application of the map T. Correspondingly, the continuous-time system describes a motion in which a particle remains forever trapped in a given region of phase space.

Often, as a parameter in the Hamiltonian (say k) is changed, discrete trapping regions can merge together—thereby qualitatively changing the nature of the long-term motion. For this change to occur, the intervening KAM trajectory must somehow disappear or dissolve at some critical value of the parameter, k_c. This Letter reports some recent work on the mechanism by which this dissolution occurs.

Focus on the particular case in which θ has the meaning of an angle divided by 2π so that (r, θ) and $(r, \theta+1)$ are identical. Different points on the KAM curve are distinguished by a "time" parameter t, so that the curve may be written as $z(t) = (r(t), \theta(t))$. In the particular case in which the curve wraps around the origin, we can consider $r(t)$ and $\theta(t) - t$ to be periodic functions of t with period 1. The Moser twist theorem[1] is then the statement that for a large class of T's one can choose $r(t)$ and $\theta(t)$ to be analytic functions of t. From this theorem, the effect of the map is simply to advance the "time" by an amount, w, called the rotation number. In symbols

$$z(t+w)=T(z(t)). \quad (1)$$

For $z(t)$ to be a continuous curve, w must be irrational.

As k passes through k_c, the KAM curve $z(t)$ dissolves. Precisely at k_c it still exists but is no longer smooth.[3] (For each w there is a different k_c.) The task of this note is to describe the nature of the bumps which appear in $z(t)$.

For specificity, focus[3] upon a particular value of w, namely the inverse of the golden mean, $w^*=(5^{1/2}-1)/2$. Then there exist two sequences of integers $q_n, p_n, n=0,1,2\ldots,$ such that $w^*q_n-p_n=(-1)^n(w^*)^{n+1}\equiv\tau_n$ is a geometrically decreasing sequence.[4] For this reason the application of the map T to $z(t)$ q_n times produces a new point on the curve very close to the old one:

$$T^{q_n}(z(t))=z(t+\tau_n). \quad (2)$$

Here q_n and p_n are respectively given by Fibonacci numbers F_n and F_{n-1} where $F_0=F_1=1$ and $F_{n+1}=F_n+F_{n-1}$. Because of this addition property $T_n=T^{q_n}$ obeys a functional recursion relation:

$$T_{n+1}(z)=T_n(T_{n-1}(z)). \quad (3)$$

In other papers[5,6] we studied T_n most specifically for the case in which T was the "standard map": $T(r,\theta)=(r',\theta')$ with $r'=r-(k/2\pi)\sin 2\pi\theta$, $\theta'=\theta+r'$. Shenker and I concluded that the recursion relation (3) had a "scaling" solution for θ close to 0 or $\frac{1}{2}$. In terms of variables θ and $u=r-F(\theta)$ (where F is an *a priori* unknown but smooth function of θ) one can represent the result of q_n steps of the mapping T upon u, θ by writing $u'=T_n^{\,u}(u,\theta)$, $\theta'=T_n^{\,\theta}(u,\theta)$. For $k=k_c$, large n, and small u and θ, our results suggest the scaling laws

$$T_n^{\,u}(u,\theta)=\beta_0^{-n}T_u^*(u\beta_0^{\,n},\theta\alpha_0^{\,n}),$$
$$T_n^{\,\theta}(u,\theta)=\beta_0^{-n}T_\theta^*(u\beta_0^{\,n},\theta\alpha_0^{\,n}). \quad (4)$$

The universal functions T_u^* and T_θ^* then describe the behavior near the dominant symmetry line[7] at $\theta=0$. Here α_0 and β_0 are universal scaling constants given by $\alpha_0=-1.414\,85\pm0.000\,03$, $\beta_0=-3.066\,86\pm0.000\,03$. Notice the similarity of the formulation to that of Feigenbaum.[8] For the KAM curve $\theta(t), u(t)$, Eq. (4) implies that for large n and small t

$$\beta_0^{\,n}u(t+\tau_n)=T_u^*(\beta_0^{\,n}u(t),\alpha_0^{\,n}\theta(t)),$$
$$\alpha_0^{\,n}\theta(t+\tau_n)=T_\theta^*(\beta_0^{\,n}u(t),\alpha_0^{\,n}\theta(t)), \quad (5)$$

where by definition $\theta(0)=0$. To satisfy Eq. (5) for all n, make a scaling assertion:

$$\alpha_0^{\,n}\theta(t)=\theta_0^*(t/\tau_n),$$
$$\beta_0^{\,n}u(t)=u_0^*(t/\beta_n), \quad (6)$$

for small t. Here the universal functions θ_0^* and u_0^* obey

$$u_0^*(t+1)=T_u^*(u_0^*(t),\theta_0^*(t)),$$
$$\theta_0^*(t+1)=T_\theta^*(u_0^*(t),\theta_0^*(t)). \quad (7)$$

The scaling hypothesis, Eq. (6), then implies that

$$\theta_0^*(t)=|t|^{x_0}\Theta_0(\log_w|t|)\,\mathrm{sgn}\,t,$$
$$u_0^*(t)=|t|^{y_0}U_0(\log_w|t|), \quad (8)$$

where $x_0=|\log_w\alpha_0|^{-1}$, $y_0=|\log_w\beta_0|^{-1}$, and $\Theta_0(x)$ is periodic with period 1 while $U_0(x+1)=-U_0(x)$. Equation (8) is then a scaling statement for the KAM curve near $\theta=0$ in which x_0 and y_0 determine the nature of the singularity at one point in this curve.

A similar behavior occurs near $\theta=\frac{1}{2}$, i.e., near $t=\frac{1}{2}$. In this region a modified form of Eq. (4) holds with different scaling indices, $\alpha_1=-1.692\,25\pm0.000\,08$ and $\beta_1=-2.564\,20\pm0.000\,01$. The analogs of T_u^* and T_θ^* were observed in Refs. 4 and 5 to depend upon n but to be periodic in n with period 3. Thus, for t near $\frac{1}{2}$ Eq. (8) is replaced by a similar statement for scaling functions:

$$\alpha_1^{\,n}\theta(t-\tfrac{1}{2})=\theta_1^*(t/\tau_n),$$
$$\beta_1^{\,n}u(t-\tfrac{1}{2})=u_1^*(t/\tau_n), \quad (9)$$

so that Eq. (8) holds once more with subscript 0 replaced by 1. Now $\Theta_1(x)$ and $U_1(x)$ are respectively even and odd under $x\to x+3$.

The analysis given here can be extended to other values of the rotation number. Let $q_0, q_1, p_1=1$ and $p_0=0$ as before. However, following, for example, Greene,[3] generalize the recursion relation for q_n and p_n to the statements

$$q_{n+1}=a_nq_n+q_{n-1},$$
$$p_{n+1}=a_np_n+p_{n-1}, \quad (10)$$

where the a_n are positive integers. Let w be the $n\to\infty$ limit of p_n/q_n. Just so long as $a_n=1$ for sufficiently large n (i.e., $n>M$), then the large-n behavior is still determined by the recursion relation (3) and if k is chosen to be the critical value for that w, i.e., $k=k_c(w)$, the critical behavior is probably exactly the same as for $w=w^*$.

One can also infer some of the dependence of $k_c(w)$ upon w. Let $a_n=1$ for $n\neq M\gg1$, $a_M=\hat{a}$, an integer greater than 1. Then the resulting rotation number is

$$w(\hat{a},M) \approx w^*\{1+(-1)^M(w^*)^{2M+1}(\hat{a}-1) \times (1+2w^*)/[2+(\hat{a}-2)w^*]\}.$$

The corresponding critical value of k can be written as $k_c(\hat{a},M)=k_c(\infty)-\epsilon$, where $k_c(\infty)$ is the critical coupling when $w=w^*$.

According to the numerical work of Shenker and myself,[5] as one progresses through cycles of length q_n for $n \leq M-1$, the residues of the resulting cycles are, for small ϵ, a function of ϵq_n. This suggests that ϵ may be viewed as a scaling field[9] which grows as $\epsilon_n \sim \epsilon q_n \sim \epsilon (w^*)^{-n}$. Then at $n=M$ the kind of recursion changes since at that step $T_{M+1}=(T_M)^{\hat{a}}\cdot T_{M-1}$. This then produces a sudden and large change in the value of the scaling field in this and the next few steps in iteration. If $\hat{a}$ is of order unity, the scaling field will change by an amount which also is of order unity. But then as one goes on many more steps in iteration T_n must settle down to its original fixed point behavior, hence showing a scaling field $\epsilon_n = 0$ for $n \gg M$. If the change near step M is of order unity, the value $\epsilon_{M-1} \sim \epsilon (w^*)^{-M}$ must also be a number of order unity which depends on $\hat{a}$. Hence, we say $\epsilon_{M-1}=\alpha\epsilon(\hat{a})$ or

$$k_c(\hat{a},M)=k_c(\infty)-(w^*)^{-M}\alpha\,\delta\epsilon(\hat{a}) \qquad (11)$$

for $M \gg 1$. Here $\delta\epsilon(\hat{a})$ is universal, but α is not.

This work was based upon a long and fruitful collaborative effort with Scott J. Shenker, which is in part reflected in our joint paper,[5] but which also includes many unpublished ideas which led up to the present paper. I have also had useful discussions with S. Aubry, J. Greene, M. Feigenbaum, and M. Widom.

[1] Radu Balescu, *Equilibrium and Non-Equilibrium Statistical Mechanics* (Wiley, New York, 1975), Appendix, gives a brief introduction to the subject of Kolmogorov-Arnold-Moser (KAM) trajectories and lists refences. See also J. Moser, *Stable and Random Motions in Dynamical Systems* (Princeton Univ. Press, Princeton, New Jersey, 1973).

[2] B. V. Chirikov, Phys. Rep. 52, 264 (1979).

[3] J. M. Greene, J. Math. Phys. 20, 1173 (1979).

[4] For large n where $q_{n+1}=q_n+q_{n-1}$ and $p_{n+1}=p_n+p_{n-1}$, q_n is always exactly of the form $q_n=\alpha(w^*)^{-n}+\beta(-w^*)^n$, as is p_n. Then the subtraction eliminates the exponentially growing terms leaving $w^*q_n-p_n \sim (-w^*)^n$.

[5] Scott J. Shenker and Leo P. Kadanoff, to be published.

[6] Leo P. Kadanoff, to be published.

[7] J. M. Greene, R. S. MacKay, F. Vivaldi, and M. J. Feigenbaum, to be published.

[8] M. Feigenbaum, J. Stat. Phys. 19, 25 (1978), and 21, 669 (1979).

[9] F. J. Wegner, *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 6, p. 8.

Classical Hopping Conduction in Random One-Dimensional Systems: Nonuniversal Limit Theorems and Quasilocalization Effects

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The $L\to\infty$ asymptotic properties of $\rho_L(g)$, the probability distribution of the classical hopping conductivity g_L corresponding to random one-dimensional systems of length L, are determined. These properties are nonuniversal, and become anomalous if the probability density $\rho(w)$ of the random near-neighbor hopping rates is such that $\int_0^\infty dw\,\rho(w)\,w^{-1}$ does not exist. The associated quasilocalization effects are discussed and their experimental observability is speculated upon.

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The transport properties of random one-dimensional systems represent a topic of high current interest, and the theoretical as well as the experimental situation is still quite controversial, in particular with respect to localization effects. For quantum systems, the zero-temperature resistance is expected[1,2] to increase exponentially with L, the length of the one-dimensional system, reflecting exponential localization of the electronic eigenstates. Analytical[2-4] and numerical[5] in-

