Eighth-Order Anomalous Magnetic Moment of the Electron

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A very preliminary value $(-0.8\pm2.5)(\alpha/\pi)^4$ is reported for the complete eighth-order QED contribution to the electron anomalous magnetic moment. The large error reflects the difficulty in evaluating the huge integrals involved and will be reduced in the future. Theory and experiment agree within 2 standard deviations. The current result enables us to determine, in a manner based solely upon elementary-particle physics, the fine-structure constant α to an accuracy of 0.08 ppm.

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The anomalous magnetic moment of the electron, $a_e=(g_e-2)/2$, has always played a central role in testing the validity of quantum electrodynamics (QED). At present the best measured values of a_e for the electron and positron are[1,2]

$$a_{e^-}{}^{\text{expt}}=1\,159\,652\,200(40)\times10^{-12},$$
$$a_{e^+}{}^{\text{expt}}=1\,159\,652\,222(50)\times10^{-12}. \qquad (1)$$

The agreement between these values affirms the the validity of the TCP theorem for the electron g factor to the level of 10^{-10}.

The QED prediction for a_e can be written as a power series in α/π,

$$a_e{}^{\text{QED}}=C_1(\alpha/\pi)+C_2(\alpha/\pi)^2+C_3(\alpha/\pi)^3+C_4(\alpha/\pi)^4+\dots \qquad (2)$$

The first three coefficients have been calculated[3]:

$$C_1=0.5,$$
$$C_2=-0.328\,478\,966\dots, \qquad (3)$$
$$C_3=1.176\,5(13).$$

If one uses the best current value[4] of the fine-structure constant

$$\alpha^{-1}=137.035\,963(15), \qquad (4)$$

the QED prediction (3) gives

$$a_e{}^{\text{QED}}=1\,159\,652\,478\times10^{-12}. \qquad (5)$$

Comparing (1) and (5) we see that the experiment and QED differ by

$$\Delta a_e=a_e{}^{\text{expt}}-a_e{}^{\text{QED}}\sim-270\times10^{-12}, \qquad (6)$$

which is nearly 7 times the experimental error quoted in (1).

In order to assess the significance of this discrepancy, one must examine the errors in (5) (including other possible contributions to a_e). The experimental uncertainty due to the value of α quoted in (4) contributes an error of 127×10^{-12} to (5). The purely computational error due to C_3 amounts to 17×10^{-12}. This results from the 21 integrals (out of 72) which are not yet known analytically. Of these, sixteen have been evaluated by a combined analytic-numerical technique (Ref. 3) and produce negligible error compared with the remaining five. The latter error will be reduced soon, either by the analtyical-numerical technique or by pushing the purely numerical integration harder. The error from the remaining terms in the series (2) is presumably of the order of

$$(\alpha/\pi)^4\sim29\times10^{-12}. \qquad (7)$$

Further effects to be taken into account include those of the muon loop, the tauon loop [both are QED contributions but are not included in (5)], the hadronic contribution, and the effect of weak interaction (we assume the standard Weinberg-Salam model)[5]:

$$a_e(\text{muon})=2.8\times10^{-12},$$
$$a_e(\text{tauon})=0.1\times10^{-12},$$
$$a_e(\text{hadronic})=1.6(2)\times10^{-12}, \qquad (8)$$
$$a_e(\text{weak})\simeq0.05\times10^{-12}.$$

Besides the uncertainties in the values (1) and (4), we thus find three possible sources for the discrepency (6): The coefficient C_4, deviation of weak interaction from the standard electroweak theory, and electron internal structure. A significant deviation of a_e(weak) from (8) would require a substantial mixing in of right-handed weak current. Better experimental limits on this will become available in a few years. The effect of possible compositeness of the electron on a_e is discussed at the end. If we assume that (6) is due entirely to the C_4 term, we obtain the "prediction"

$$C_4 \sim -9.3, \qquad (9)$$

a rather large value. In any event, as is obvious from (7) and (9), it is no longer possible to put QED to a stringent test commensurate with the very precise measurements available unless we know the sign and magnitude of C_4.

It is for this reason that we decided to calculate C_4. In the absence of a reliable quick method, we had no choice but to calculate by brute force the values of all 891 Feynman diagrams that contribute to C_4. However, a substantial simplification was achieved by a method developed earlier[6] which enables us to combine several amplitudes into one and reduce the number of integrals to be evaluated to about 120 (including integrals of lower-order amplitudes needed in the renormalization scheme).

The diagrams fall naturally into the following five groups, each of which consists of one or more gauge-invariant sets.

Group I.—Second-order vertex diagram containing vacuum polarization loops of second, fourth, and sixth orders. This group consists of 25 diagrams (10 integrals).

Group II.—Fourth-order vertex diagrams containing vacuum polarization loops of second and fourth orders. This group contains 54 diagrams (8 integrals).

Group III.—Sixth-order vertex diagrams containing a vacuum polarization loop of second order. There are 150 diagrams (8 integrals) in this group.

Group IV.—Vertex diagrams containing a photon-photon scattering subdiagram with further radiative corrections of various kinds. This group consists of 144 diagrams (13 integrals).

Group V.—Vertex diagrams containing no vacuum polarization loop. This group is comprised of 518 diagrams (47 integrals).

Integrands were generated by the algebraic program SCHOONSCHIP.[7] A typical integrand consists of a rational function of up to 15 000 terms, each term being a product of up to nine rational functions of Feynman parameters. Integrals were renormalized by the scheme developed in Ref. 6. The integrations, over hypercubes of up to ten dimensions, were carried out largely by the adaptive Monte Carlo subroutine RIWIAD.[8] The routine VEGAS[9] was also used in some instances. Both give comparable accuracy for these large integrals.

The results for the five groups are (see Kinoshita and Lindquist[10] for a detailed description of the calculation of each group)

$$\begin{aligned} C_{\rm I} &= 0.0766(6), \\ C_{\rm II} &= -0.5238(10), \\ C_{\rm III} &= 1.419(16), \\ C_{\rm IV} &= -0.78(48), \\ C_{\rm V} &= -1.0(2.4), \end{aligned} \qquad (10)$$

giving a total contribution

$$C_4 = -0.8(2.5). \qquad (11)$$

The errors for $C_{\rm I}$ through $C_{\rm IV}$ are 90% confidence limits (CL) as estimated by the integration routine. The values $C_{\rm IV}$ and $C_{\rm V}$ are very tentative as adequate sampling of the integration domain has not yet been achieved (excessive computing time required on scalar machines) for many of the integrals in these groups. The quoted error for $C_{\rm V}$ is again an estimate of a 90% CL; however, in cases where sampling was clearly inadequate, we have arbitrarily multiplied the 90%-CL error estimated by the integration routine by a factor of 1.2 or 1.4 to represent our mistrust of the results. Note that errors of individual integrals are less than 10% in most cases and the small central value of $C_{\rm V}$ relative to its error results from cancellation of large terms.

The main significance of our result is the establishment of finite (though rather soft as yet) bounds for C_4. It appears that the "prediction" (9) is not borne out by our calculation. On the other hand, our $C_{\rm V}$ is consistent with $-15.6/16$ predicted from the study of large orders of perturbation theory.[11] With the result (11), and including (8), the value for a_e through eighth-order QED is

$$a_e^{\rm theor} = 1\,159\,652\,460(127)(75) \times 10^{-12}, \qquad (12)$$

consistent, at the 2-standard-deviation level, with (1). Here the errors 127×10^{-12} and 75×10^{-12} are due to those of α in (4) and theory, respectively. The error in (11) will be reduced substantially in the future by taking advantage of array processors and the inherent vectorizability of our calculation.

An important by-product of the study of a_e is the determination of α solely within the theoretical framework of elementary-particle physics. We give below α determined from (1), (3), (8), and (11). For comparison we also list α determined from the muonium hfs,[12] the ac Josephson effect,[4] and the new method utilizing the quantum

Hall resistance[13]:

$$\begin{aligned}\alpha^{-1}(a_e) &= 137.035\,993(5)(9)\,,\\ \alpha^{-1}(\text{muonium hfs}) &= 137.035\,989(3)(47)\,,\\ \alpha^{-1}(\text{ac Josephson}) &= 137.035\,963(15)(\,?)\,,\\ \alpha^{-1}(\text{quantum Hall}) &= 137.035\,968(23)(\,?)\,.\end{aligned} \tag{13}$$

Here the first errors are experimental and the second theoretical.[14] The values of α^{-1} in (13) are in reasonable agreement with each other. However, further reduction of errors could reveal significant discrepancies.

The value of α^{-1}(ac Josephson) is based on the very accurate measurement of $2e/h$ (0.03 ppm at present) by the ac Josephson effect. The error in α^{-1} comes mainly from the measurement of proton gyromagnetic ratio which is needed in converting $2e/h$ to α. Works are under way to reduce this error substantially. On the other hand, we find little discussion in the literature of theoretical errors in the $2e/h$ measurement except that it appears not to be susceptible to higher-order QED corrections.[15] To emphasize this we have put a question mark for the theoretical error of α^{-1}(ac Josephson) in (13). In view of its great importance we strongly urge careful assessment of errors which might be present in the theory (ranging from phenomenological to fundamental) of the Josephson effect.

The new method of determining α discovered by von Klitzing, Dorda, and Pepper[13] has already achieved an accuracy comparable to that of the ac Josephson effect. Thus we should like to see similar questions about its theoretical foundation answered in order to establish it as a viable method for high-precision determination of α.

Finally, let us consider a possible composite structure of the electron, which is being investigated in various models.[16] If we assume that the contribution of a constituent particle of mass M to a_e is given by[17]

$$\Delta a_e = O(m_e/M) = O(m_e R_e)\,, \tag{14}$$

where R_e is the effective radius of the electron, we find, using the current experimental limit $R_e \lesssim 10^{-16}$ cm, that

$$|\Delta a_e| \lesssim 3\times 10^{-6}\,, \tag{15}$$

which is far too crude. It has been suggested instead[17] that the difference $a_e^{\text{expt}} - a_e^{\text{theor}}$ should be regarded as an upper bound for possible structure effect. If one adopts this viewpoint one obtains a very stringent lower bound for M[18]:

$$M \gtrsim 10^6 \text{ GeV or } R_e \lesssim 10^{-20} \text{ cm}\,. \tag{16}$$

Unfortunately this assumes implicitly that there is no theoretical error in the determination of α in (6), which is far from obvious as was noted above.

A way to circumvent this is to compare $\alpha(a_e)$ and α(muonium hfs). With the impending improvement[19] in the theoretical error of α(muonium hfs), which happens to be extremely insensitive to lepton internal structure,[20] it will be possible to give more reliable bounds for the mass of constituent particles of the electron in a manner independent of theoretical uncertainties in α(ac Josephson).

An excellent agreement between $\alpha(a_e)$ and α(muonium hfs) will not only test the internal consistency of QED and give a more stringent limit on the internal structure of the electron but also provide a strong challenge to the theoretical basis of condensed-matter measurements of α.

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[2]The quantity enclosed in parentheses represents the uncertainty in the final digits of numerical value.

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Experimental Limits on Neutrino Oscillations

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A search for neutrino oscillations in a wide-band neutrino beam at Fermilab with use of the 15-ft bubble chamber is reported. No evidence is found for neutrino oscillations and upper limits are set on the mixing angles and neutrino mass differences in the transitions $\nu_\mu \to \nu_e$, $\nu_\mu \to \nu_\tau$, and $\nu_e \to \nu_{\sim e}$, where $\sim e$ denotes "not *e*."

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Various authors have investigated[1] the possibility of neutrino oscillations, i.e., the time-dependent mixing between different types of neutrinos. These oscillations can only occur if there is a nonzero mass difference between the neutrinos involved and the lepton numbers of the neutrinos are not rigorously conserved. With three or more neutrino types, the situation is quite complex, and depends on many parameters. In this paper, we consider only oscillations between two types of neutrinos at a time. In this case, the observed neutrino types, say ν_α and ν_β, are quantum mechanical mixtures of the neutrino mass eigenstates, ν_1 and ν_2:

$$\nu_\alpha = \cos\theta\nu_1 + \sin\theta\nu_2,$$

$$\nu_\beta = -\sin\theta\nu_1 + \cos\theta\nu_2,$$

where θ is the mixing angle between the two types of neutrinos. The probability of the appearance of a neutrino ν_β, when initially a neutrino ν_α was created, is

$$P(\nu_\alpha \to \nu_\beta) = \sin^2(2\theta)\sin^2(1.27\Delta m^2 l/E),$$

where $\Delta m^2 = m_1^2 - m_2^2$ is in units of electronvolts squared, E is the neutrino energy in megaelec-

