Description of Pulse Dynamics in Lorentz Media in Terms of the Energy Velocity and Attenuation of Time-Harmonic Waves

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(Received 12 June 1981)

The first detailed physical explanation of the dynamics of short pulses in a dispersive system that has loss is presented. The result provides a simple algorithm for predicting pulse behavior without the need for asymptotic analysis. It was derived specifically for electromagnetic pulses in Lorentz media, making it applicable to low-energy pulses in resonant media, pulses in plasmas and dielectrics, remote ionospheric measurements, and many pulsed laser systems. It appears likely, however, that the result applies to even more general wave systems.

PACS numbers: 41.10.Hv

There is a simple physical description available for the dynamics of a plane-wave pulse after it has traveled sufficiently far in a dispersive system provided the loss in the system can be neglected.[1] There is no such description known, however, for any dispersive system in which loss is important.[2] The lossy system that is best understood is the Lorentz medium. In that system, the dynamics of pulses have been determined in detail by a very involved asymptotic analysis.[3] However, there has been no physical explanation of the results; it is not known why (in physical terms) the absorption in the medium affects the evolution of the field amplitude and instantaneous frequency in the way it does. Because of this, it has not been possible to apply the known results for the Lorentz medium to surmise the evolution of pulses in other lossy dispersive systems.

In this paper, we present a new mathematical result that provides a physical explanation of the evolution of the field amplitude and instaneous frequency in a Lorentz medium. The result also provides a simple algorithm for predicting pulse dynamics without the need for complicated asymptotic analysis. The algorithm is directly applicable to a wide range of physical systems such as low-energy pulses in resonant media, pulses in plasmas and dielectrics, picosecond laser systems, and remote ionospheric probes since the Lorentz medium provides an accurate model for these systems. Moreover, because of the physical nature of the result, it appears likely that it can be used to describe and predict the dynamics of pulses in other lossy dispersive systems such as linear elastic, electromagnetic, gravity, and atmospheric waves in various media and guiding structures.

Consider a plane electromagnetic wave with real electric field $E(z, t)$ linearly polarized along the x axis and traveling in the positive z direction through a linear, homogeneous, isotropic, temporally dispersive medium occupying the half-space $z>0$. The field can be expressed in a Laplace representation as

$$E(z, t) = \int_{ia-\infty}^{ia+\infty} E(z, \omega) e^{-i\omega t} d\omega, \qquad (1)$$

where a is an arbitrary positive constant and the spectral amplitude $E(z, \omega)$ satisfies the scalar Helmholtz equation $[\nabla^2 + k^2(\omega)]E(z, \omega) = 0$. The complex propagation constant $k(\omega)$ is given in terms of the complex index of refraction $n(\omega)$ by $k(\omega) = \omega n(\omega)/c$ where c is the speed of light in vacuo. For a Lorentz medium with a single resonance frequency, $n(\omega)$ is given by

$$n(\omega) = \left[1 - \frac{b^2}{\omega^2 - \omega_0^2 + 2\delta i\omega}\right]^{1/2}, \qquad (2)$$

where b, δ, and ω_0 are positive constants. For our numerical calculations, we have used the same parameter values as used by Brillouin,[3] i.e., $b^2 = 2\times 10^{33}/\text{sec}^2$, $\delta = 2.8\times 10^{15}/\text{sec}$, and $\omega_0 = 4\times 10^{16}/\text{sec}$. This corresponds to a highly absorbing dielectric.

Let the field satisfy the boundary value $E(0, t) = f(t)$ where $f(t)$ is a real function that satisfies $f(t) = 0$ for $t<0$. The special case of $f(t) = \sin(\omega_c t)$ for $t>0$ is a classical problem first treated by Sommerfeld and Brillouin.[3] We are concerned here with $f(t)$ that is zero after some finite time. A case of primary interest is the δ-function pulse having $f(t)=\delta(t)$. In that case, $E(z,t)$ for fixed z is the impulse response of the medium.

By applying the approach used by Brillouin,[3] we find that the asymptotic approximation of $E(z,t)$ valid as $z\to\infty$ with fixed $\theta=ct/z$ is given by a constant K times the real part of the function[4]

$$A(z,t)=\frac{F(\omega_s)}{[-z\varphi^{(2)}(\omega_s)]^{1/2}}\exp\left[\frac{z}{c}\varphi(\omega_s)\right], \qquad (3)$$

where

$$F(\omega)=(2\pi)^{-1}\int_0^\infty f(t)e^{i\omega t}dt, \qquad (4)$$

and $\varphi(\omega)=i\omega[n(\omega)-\theta]$. The quantity $\varphi^{(2)}(\omega_s)$ is the second derivative of φ evaluated at ω_s. The frequency ω_s is a saddle point of the phase function $\varphi(\omega)$. For θ in a range $1<\theta<\theta_{SB}$, the constant K is $2(2\pi c)^{1/2}$ and ω_s is in the fourth quadrant of the complex ω plane, slightly below the real axis and moving towards the left with increasing θ. For θ in another range $\theta_{SB}<\theta<\theta_1$, $K=(2\pi c)^{1/2}$ and ω_s is on the imaginary axis moving down with increasing θ. For $\theta>\theta_1$, $K=2(2\pi c)^{1/2}$ and ω_s is in the fourth quadrant slightly below the real axis and moving towards the right-hand side with increasing θ. Approximate expressions for ω_s, θ_{SB}, and θ_1 are given in Ref. 4. For Brillouin's parameter values, $\theta_{SB}=1.295$ and $\theta_1=1.503$. The discontinuities in this asymptotic approximation at $\theta=\theta_{SB}$ and θ_1 have been eliminated by a uniform asymptotic approach in Ref. 4 but the results are not needed here.

The above results parallel those obtained by Brillouin.[3] By obtaining much more accurate approximations for the locations of the saddle points, however, we have found a new result that could not have been discovered using Brillouin's expressions. Let Z' and Z'', respectively, represent the real and imaginary parts of any complex number Z. Then, we have found that for $1<\theta<\theta_{SB}$ and for $\theta_1<\theta<\theta_{max}$ with θ_{max} defined below, the real part of the function occurring in the exponent in (3) satisfies

$$\varphi'(\omega_s)\approx\varphi'(\omega_E)\equiv-\omega_E n''(\omega_E), \qquad (5)$$

where ω_E is a real frequency that satisfies

$$V_E(\omega_E)\equiv\frac{c}{n'(\omega_E)+n''(\omega_E)\omega_E/\delta}=\frac{c}{\theta}\equiv\frac{z}{t}. \qquad (6)$$

This result was obtained by deriving approximate analytic expressions for $\varphi'(\omega_s)$ and $\varphi'(\omega_E)$ and showing them to be equivalent. Moreover, it was checked for Brillouin's parameters by using exact expressions evaluated with numerical techniques. The procedure used was to calculate two different values of θ for each real frequency ω of interest. The first value of θ was $c/V_E(\omega)$ whereas the second was the value of θ at which $\varphi'(\omega_s)=\varphi'(\omega)$. These values were found to agree to within 6% for all θ in the range of interest.

The quantity $V_E(\omega)$ is the velocity of energy of a time-harmonic wave with angular frequency ω, as determined by the ratio of the Poynting vector to the density of energy including both the energy of the field and the energy stored in the medium.[5] The quantity θ_{max} is defined to be the value of θ given by (6) for $\omega_E=\omega_{max}$ where $1/V_E(\omega)$ has its maximum value at ω_{max}. When there is more than one value of ω_E that satisfies (6) for a given θ, (5) is satisfied with the value of ω_E that yields the smallest value of $\varphi'(\omega_E)$.

One important consequence of (5) is that the attenuation of $A(z,t)$ with increasing z and constant θ can be determined without requiring the knowledge of the location of the saddle point ω_s. Since ω_s is constant for fixed θ, Eq. (3) shows that the attenuation of $A(z,t)$ with increasing z and fixed θ is determined by $\varphi'(\omega_s)$. Hence, the attenuation can be found by using (6) to evaluate ω_E and substituting the result into (5). Since fixed θ implies that the point of observation is moving with velocity c/θ, it is seen that *the attenuation of $A(z,t)$ is the same as that of the time-harmonic wave with the least attenuation that has energy velocity equal to the velocity of the point of observation.*

For z sufficiently large, (3) remains valid even when z and t vary in a way so that θ does not remain constant. In that case, ω_s and ω_E change as z and t change. For $1<\theta<\theta_{SB}$ and $\theta_1<\theta<\theta_{max}$, ω_s' is approximately equal to ω_E. (This approximation gradually deteriorates towards the tail end of the pulse at $\theta=\theta_{max}$, where the error is about 10% for Brillouin's parameters.) Moreover, ω_s'' is very small compared to ω_s' for θ not too close to θ_1. Hence, ω_s can be replaced by ω_E in the slowly varying functions in (3). Since z is large, the most rapidly varying function is

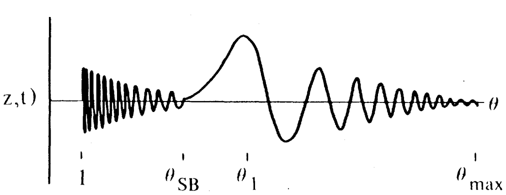

FIG. 1. Asymptotic behavior of a δ-function pulse for fixed large z.

the exponential. The attenuation is already expressed in terms of ω_E in (6). The oscillations are determined by the exponent

$$iz\varphi''(\omega)/c = i[-\omega' t + zn'(\omega)\omega'/c - zn''(\omega)\omega''/c]$$

evaluated at $\omega=\omega_s$. If we replace ω_s by ω_E in the index of refraction terms and set $\omega_s'=\omega_E$ elsewhere in this expression, it becomes

$$iz\varphi''(\omega_s)/c \approx i[-\omega_E t + zk'(\omega_E) - zn''(\omega_E)\omega_s''/c].$$

Combining this result with (5) shows that (3) can be approximated by

$$A(z,t)\approx \frac{F(\omega_E)}{[-z\varphi^{(2)}(\omega_E)]^{1/2}}\exp\{i[-\omega_s''n''(\omega_E)z/c]\}\exp\{i[k(\omega_E)z-\omega_E t]\}. \tag{7}$$

Equation (7) displays $A(z,t)$ as a modulated wave with angular frequency ω_E and propagation constant $k(\omega_E)$. For fixed z, the wave is chirped since ω_E is a function of time according to (6). The amplitude is modulated by the factor in front of the exponential but is primarily determined by the change in the attenuation coefficient $k''(\omega_E)$ as ω_E changes with time. For fixed t, the z behavior is primarily determined by the propagation constant $k(\omega_E)$ which changes with z because ω_E changes according to (6). If δ is large enough, the wavelength is shifted slightly due to the first exponential term in (7) when ω_E is in the absorption band. Apart from this small shift, the properties of the pulse can be obtained from (7) with the knowledge of ω_E without the need to know ω_s. *The field is dominated by a single real frequency at each space-time point. That frequency ω_E is the frequency of the time-harmonic wave with the least attenuation that has energy velocity equal to z/t.*

To illustrate the utility of (7) to obtain the primary features of the pulse, consider the δ-function pulse. Then, $F(\omega)=1$. Figure 1 is a schematic drawing of the behavior of a δ-function pulse as determined by (3). The high-frequency field at the beginning corresponds to the first precursor found by Sommerfeld and the larger-amplitude, low-frequency field corresponds to the second precursor found by Brillouin. If z is taken to be constant, then θ is proportional to t and Fig. 1 shows the time behavior of $E(z,t)$. The primary features of this behavior can be obtained from the behavior of the second exponential term in (7) with the knowledge of ω_E. A plot of ω_E vs θ is given in Fig. 2 for Brillouin's parameters calculated by using (6) to evaluate θ for various values of ω_E. Values of θ_{max} and ω_{max} follow directly from the plot. The value of θ_1 does not show directly on the plot but it is very close to θ_0 which is the value of θ corresponding to $\omega_E=0$. Unfortunately, θ_{SB} cannot be obtained without the knowledge of ω_s.

From Fig. 2, we see that for $1<\theta<\theta_{SB}$, ω_E is large and decreasing with increasing θ. Since ω_E is moving towards ω_{max} which is in the center of the absorption band, $k''(\omega_E)$ is increasing. This leads to a high-frequency field with amplitude and frequency decreasing as θ increases from 1 to θ_{SB} in agreement with Fig. 1. For $\theta_1<\theta<\theta_{max}$, there are two possible values of ω_E. The smaller value leads to the least attenuation and hence that frequency dominates the field. Consequently at $\theta=\theta_1$, we have $\omega_E\approx 0$ leading to very low attenuation and a large-amplitude, low-frequency field. For increasing θ, ω_E increases towards ω_{max} leading to increasing attenuation. Hence, the frequency of the field increases and the amplitude decreases as θ increases, again in agreement with Fig. 1. By the time θ reaches θ_{max}, the amplitude is very small and the pulse can be considered to have passed. As a result, we have obtained the behavior of the pulse as shown in Fig. 1 for all θ of interest except during the short period when the transition between the two precursors is taking place. During that peri-

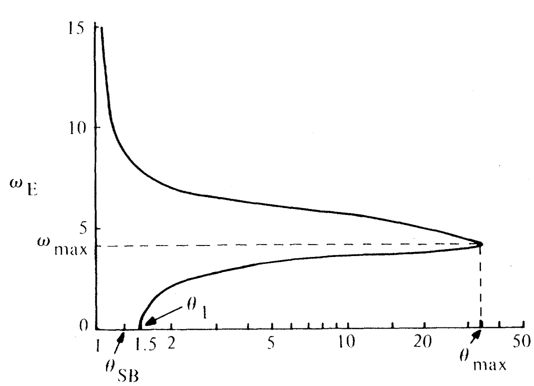

FIG. 2. Angular frequency ω_E (in units of 10^{16}/sec) of a time-harmonic wave having energy velocity c/θ.

od, the field grows without oscillation and is not dominated by any real frequency.

The above description can be extended to treat transients that have a steady-state component such as the step-modulated time-harmonic field discussed by Brillouin.[3] In that case, $F(\omega)$ has a pole on the real axis at the signal frequency ω_c. The correct dynamics are obtained for such a case if a time-harmonic component with angular frequency ω_c is added to the field obtained by the above description for all $\theta > \theta_c$ where θ_c is the value of θ at which $\omega_E = \omega_c$. The field is dominated by the time-harmonic signal for all $\theta > \theta_c$, such that $k''(\omega_E) > k''(\omega_c)$. For all other θ, the time-harmonic component is negligible compared to the precursor field. Although the dynamics of the step-modulated time-harmonic signal obtained by this prescription are very different from the classical results of Brillouin,[3] they are identical to those obtained by a more accurate asymptotic approach.[6]

The description of the field we have presented is the natural extention to lossy dispersive systems of a description that has gradually emerged over the last few decades to explain the dynamics of pulses in lossless dispersive systems. The method of stationary phase has been used to show that, for rather general waves in rather general lossless dispersive systems, the field is dominated by a single real frequency at each space-time point.[1] That frequency is the one having group velocity equal to z/t. It has also been established that, under very general conditions, the energy velocity of a monochromatic wave is equal to its group velocity in lossless dispersive systems.[7] Combining these two results shows that the energy-velocity description of pulse dynamics applies to very general waves (elastic, electromagnetic, gravity, atmospheric, etc.) in very general dispersive systems (dispersive media, guiding structures, etc.) provided that the loss is negligible. (For example, see Tolstoy[1] and Lighthill.[7]) The result presented in this paper generalizes the above description to apply to an important class of dispersive systems in which loss is important. This generalization makes it appear likely that the energy-velocity description presented in this paper applies to general waves in general lossy dispersive systems as well as in lossless systems. Further research is necessary to test this hypothesis.

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