Analytic Linearization, Hamiltonian Formalism, and Infinite Sequences of Constants of Motion for the Burgers Equation

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The Burgers equation is linearized on the Schwartz space S. It follows that the Burgers equation can be defined by a Hamiltonian formalism and that it has an infinity of time-independent constants of motion in involution.

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The Burgers equation[1] in two space-time dimensions,

$$\partial u(t,x)/\partial t = \partial^2 u(t,x)/\partial x^2 + 2u(t,x)\partial u(t,x)/\partial x, \qquad x,t,u(t,x) \in R, \tag{1}$$

is often given as an example of a nonlinear differential equation which cannot be defined by a Hamiltonian formalism and which does not have an infinity of constants of motion.[2] It is proved in this note that the Burgers equation (which is dissipative and has no solitons) can be defined by a Hamiltonian system and that it has an infinity of constants of motion. The existence of sufficiently many constants of motion forces in general the solution of an evolution equation to rest on a "submanifold" of smooth initial conditions. This facilitates the integration of the equation and helps to determine physical properties of the solutions such as the development of shock waves.

Consider Eq. (1) in its abstract form on S (the Schwartz space of rapidly decreasing test functions from R to R),

$$du(t)/dt = \partial^2 u(t) + 2u(t)\partial u(t) \equiv \hat{T}_1(u(t)), \qquad u(0) = u_0 \in S, \tag{2}$$

where $[\partial v](x) = \partial v(x)/\partial x$. A representation $\hat{T}$ with respect to the vector field bracket of the two-dimensional commutative Lie algebra t_2 (space-time translation) on vector fields on S is defined by

$$t_2 \ni (a,b) \xrightarrow{\hat{T}} a\hat{T}_0 + b\hat{T}_1, \tag{3}$$

where $\hat{T}_0(u) = \partial u$ is the generator of space translation. I prove that the Lie algebra representation $\hat{T}$ is formally linearizable on S,[3] and that the intertwining formal power series defines an entire function $\hat{A}$,[4] which is analytically invertible. (This linearization is not identical to the Cole-Hopf transformation, but it has a similar structure.) The representation $\hat{T}$ turns out to be integrable to a representation $\hat{U}$ of the semigroup $R \times R_+$ on S, where R_+ is the set of all nonnegative reals. In particular this implies that Eq. (2) has a solution $u(t) \in S$, $t \in R_+$ for any initial condition $u(0) \in S$.

The function $\hat{A}$ extends to various spaces larger than S, such as the space L^1 of absolutely integrable functions. If $u_0 \in L^1$ then $u(t) \in C_0^\infty(R)$ for each $t > 0$. Thus the Burgers equation is smoothing out "singular" initial conditions.

It is easy to construct (without any reference to some Hamiltonian formalism) an infinity of constants of motion b_n for the heat equation on S. The above mentioned results allow then to find such quantities $B_n = b_n \circ \hat{A}$ for the Burgers equation. The nonlinear superposition principle and hierarchy of higher-order Burgers equations are recovered in a similar way.

Intuitively, one cannot hope to find a Hamiltonian formalism with canonical coordinates for the heat equation on S, where the equation cannot be integrated backward in time. However, I construct a family of Hamiltonian formalisms (each expressible in canonical coordinates) for the heat equation on the space $S_{1/2}$ (of functions bounded, together with their derivatives, by Gaussian functions). The heat equation can be integrated for each time on this space. The pullback by $\hat{A}$ gives then a family of Hamiltonian formalisms for the Burgers equation, each of them being completely integrable.

(1) *Linearization.*—Introduce $T_0^{\,1}(u) = \partial u$, $T_1^{\,1}(u) = \partial^2 u$, $T_1^{\,2}(u_1,u_2) = u_1\partial u_2 + u_2 \partial u_1$, and $T_i^{\,j} = 0$ otherwise for $j \geq 1$, $i = 0, 1$, $u, u_1, u_2 \in S$. Further let $A^n \in L_s(\hat{\otimes}^n S, S)$ (n-linear symmetric continuous mappings from S into S), let $F(S)$ be the space of formal power series from S to S, and let

$$B * C = \sum_{\substack{n \geq 1 \\ 1 \leq p \leq n}} B^p \Big(\sum_{0 \leq q \leq p-1} I_q \otimes C^{n-p+1} \otimes I_{p-q-1} \Big) \sigma_n$$

for $B, C \in F(S)$. Here I_q is the identity mapping on $\hat{\otimes}^q S$ and T_n is the normalized symmetrization operator. The product $B * C$ is the functional derivative of B in the direction C if B and C converge. The bracket $[\ ,\ ]_*$ is defined by $[B, C]_*$

$=B*C-C*B$.

The representation (with respect to $[\ ,\ ]_*$) $t_2 \ni (a,b) \to aT_0 + bT_1 \in F(S)$, with $T_i = \sum_{n\geq 1} T_i{}^n$, is formally linearizable[3] iff there exists $A \in F(S)$ such that

$$A * T_i = T_i{}^1 A, \quad i=0,1. \tag{4}$$

Let us introduce

$$[A^n(u_1,\ldots,u_n)](x) = \frac{1}{n!}\sum_{k=1}^n u_k(x) \prod_{l\neq k} \partial^{-1} u_l(x) \tag{5}$$

and $\hat{A}^n(u) = \hat{A}(u,u,\ldots,u)$, where $u,u_1,\ldots,u_n \in S$ and $\partial^{-1}u(x) = \int_{-\infty}^x u(\zeta)d\zeta$.

Proposition 1.—The power series $A = \sum_{n\geq 1} A^n$ formally linearizes T and $\sum_{n\geq 1}\hat{A}^n$ converges to the entire function $u \to \hat{A}(u) = \partial \exp(\partial^{-1}u)$. The inverse $\hat{A}^{-1}(u) = u(1+\partial^{-1}u)^{-1}$ is analytic on the image $\hat{A}[S]$, where

$$\hat{A}[S] = \{u \in S \mid 1 + \textstyle\int_{-\infty}^x u(\zeta)d\zeta > 0, \forall x \in R \cup \{\infty\}\}$$

is open in S.

Proof.—It is straightforward to check that (5) is a particular solution of (4).

We define a complete set of seminorms on S by

$$p_N(u) = \sup_{\substack{x\in R\\ 0\leq \alpha \leq N}} |(1+|x|)^N \partial^\alpha u(x)|, \quad N \geq 0.$$

The space $C_R{}^\infty$ is defined as the subset of $C^\infty(R)$ that satisfies

$$\|f\|_N = \sup_{\substack{x\in R\\ 0\leq \alpha \leq N}} |\partial^\alpha f(x)| < \infty, \quad N\geq 0.$$

One verifies that

$$p_N(uf^M) \leq (2M)^N p_N(u) \|f\|_N{}^N \|f\|_0{}^{M-N}, \qquad M \geq N \geq 0, \tag{6}$$

for $u \in S$, $f \in C_R{}^\infty$. It follows from (6) that

$$\sum_{k=M+1}^\infty p_N(\hat{A}^k(u)) \leq \sum_{k=M+1}^\infty \frac{[2(k-1)]^N}{(k-1)!} p_N(u) \|\partial^{-1}u\|_N{}^N \|\partial^{-1}u\|_0{}^{k-N-1} < \infty;$$

so $\hat{A}$ is entire and

$$\hat{A}(u) = \partial \exp(\partial^{-1}u). \tag{7}$$

Equation (7) gives the inverse ($\hat{A}$ is injective)

$$\hat{A}^{-1}(u) = u(1+\partial^{-1}u)^{-1}. \tag{8}$$

Equations (7) and (8) give the expression for $\hat{A}[S]$. Let $v \in \hat{A}[S]$; then $v + \varphi \in \hat{A}[S]$ if $p_2(\varphi)$ is sufficiently small, so that $\hat{A}[S]$ is open. Equations (6) and (8) give that $\hat{A}^{-1}$ is analytic, Q.E.D.

The convexity of $\hat{A}[S]$ gives the nonlinear superposition principle. The existence of a global solution $u(t) \in S$, $t \geq 0$, for each initial condition $u_0 \in S$ is ensured by the following:

Corollary 2.—The Lie algebra representation $\hat{T}$ is integrable to a C^∞ semigroup representation

$$(R\times R_+)\times S \ni (a,b,u) \to [\hat{U}(a,b)](u) \in S.$$

Proof.—T^1 is integrable to a C^∞ representation U^1 on S of $R\times R_+$. Further, the open set $\hat{A}[S]$ is mapped into itself by U^1. Hence $\hat{U} = \hat{A}^{-1} \circ U^1 \circ \hat{A}$ integrates $\hat{T}$, Q.E.D.

The commutative Lie algebra of the higher-order Burgers equations is given by $Y_n = d[\hat{A}^{-1}] \cdot X_n \circ \hat{A}$, $n = 1,2,\ldots$, where $X_n(u) = \partial^n u$ and d is the functional derivative. Explicitly the polynomial Y_n is

$$[Y_n(u)](x) = \exp[-\textstyle\int_{-\infty}^x u(\zeta)d\zeta] \displaystyle\sum_{k=0}^{n-1} \binom{n}{k} (\partial^{n-k}u)(x) \partial^k \exp[\textstyle\int_{-\infty}^x u(\zeta)d\zeta]. \tag{9}$$

We now turn to the question of constants of motion for the heat equation on S. Let $p: R^2 \to C$ be a harmonic polynomial. Then one sees that $P(u) = \int dx\, dg\, p(x,y) u(x) u(g)$ is a constant of motion for the heat equation on S. In particular, define the moments

$$b_n(u) = 2\pi)^{-1} 2^{-n/2} (i-1)^n \int dx\, dy\, (x+iy)^n u(x) u(y), \tag{10}$$

$n = 0,1,\ldots, u \in S$, and define $B_n = b_n \circ \hat{A}$. The functions B_n are then constants of motion for the Burgers equation and are explicitly given by

$$B_n(u) = (2\pi)^{-1} 2^{-n/2} \sum_{l=0}^n \binom{n}{l} (i-1)^l (-1-i)^{n-l} \times \int_{-\infty}^\infty dx [x^l u(x) \exp \textstyle\int_{-\infty}^x u(\zeta)d\zeta] \int_{-\infty}^\infty dx [x^{n-l} u(x) \exp \int_{-\infty}^x u(\zeta) d\zeta], \quad n = 0,1,\ldots, u \in S. \tag{11}$$

The functions B_n are local in the velocity potential $\partial^{-1}u$, as they are finite sums and products of integrals of densities in $\partial^{-1}u$.

Proposition 3.—$\{B_n\}_{n=1}^{\infty}$ is a set of time-independent, C^∞ constants of motion for the Burgers equation on S. The set $\{B_n\}_{n=N}^{\infty}$ is independent at u (in the sense of analytic mechanics) iff u is not of the form $u=\partial^{N+1}v/(1+\partial^N v)$, $v\in S$, $\partial^N v \in A[S]$, i.e.,

$$\sum_{n=N}^{M} k_n dB_n(u)=0, \quad k_n\in C, \quad N\leq M<\infty$$

implies $k_n=0$, $N=0,1,\ldots$. The proof is not difficult and I therefore omit it here. One proves with the help of proposition 3 that the points where $\{C_n\}_{n=N}^{\infty}$ is dependent for each N is nowhere dense in S.

(2) *Hamiltonian formalism.*—Let S_α, $1>\alpha>0$,[5] be the space of all complex valued C^∞ functions φ such that

$$|\partial^k\varphi(x)|\leq C_k(\varphi)\exp[-a(\varphi)|x|^{1/\alpha}], \quad k=0,1,\ldots$$

and let S^β, $1>\beta>0$,[5] be the space of all entire functions φ such that

$$|x^k\varphi(x+iy)|\leq C_k'(\varphi)\exp[a'(\varphi)|y|^{1/(1-\beta)}], \quad k=0,1,\ldots,$$

where $C(\varphi)$, $a(\varphi)$, $C'(\varphi)$, and $a'(\varphi)$ are some positive numbers. One has $\tilde{S}_\alpha=S^\alpha$,[5] where the Fourier-Laplace transformation is defined by

$$\tilde{\varphi}(x+iy)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}ds\exp[is(x+iy)]\varphi(S), \quad \varphi\in S_\alpha.$$

(S_α will from now on stand for a real-valued function and S^α for its image $\tilde{S}_\alpha$.)

The Burgers equation is integrable for each time $t\in R$ in $S_{1/2}$ as this is the case for the heat equation and as $\hat{A}[S_{1/2}]\subset S_{1/2}$. The existence of Hamiltonian formalisms for the heat equation is given by proposition 4.

Proposition 4.—Let $f\in S_\alpha$, $0<\alpha<\frac{1}{2}$, and $f\neq 0$. Then the form ω_f: $S_{1/2}\times S_{1/2}\rightarrow R$ defined by

$$\omega_f(\varphi,\psi)=2\mathrm{Im}\int_{-\infty}^{\infty}ds\, f(s)\tilde{\varphi}^*(S(1+i)/\sqrt{2})\tilde{\psi}(S(1+i)/\sqrt{2})$$
$$\{\equiv(2\pi)^{-1/2}\int dx\,dy\,i^{-1}[\tilde{f}((x-ig)(1-i)/\sqrt{2})-\tilde{f}((y-ix)(1-i)/\sqrt{2})]\varphi(x)\psi(y)\}$$

is a weakly nondegenerate constant symplectic form,[6] for which $T_2^{\,1}$ is a symplectic vector field, i.e., $L_{\partial^2}\omega_f=0$, where L is the Lie derivative. (The asterisk denotes complex conjugation.)

Proof.—(i) ω_f is antisymmetric and constant. (ii) $L_{\partial^2}\omega_f(\varphi,\psi)=\omega_f(\partial^2\varphi,\psi)+\omega_f(\varphi,\partial^2\psi)=0$. (iii) ω_f is weakly nondegenerate, i.e., $\omega_f(\varphi,\psi)=0\ \forall\ \psi\in S_{1/2}\rightarrow\varphi=0$; Q.E.D. We can construct canonical coordinates as follows: Let $R: u\rightarrow(p,q)$ be the transformation

$$p(s)=|\tilde{u}(s(1+i)/\sqrt{2})|^2 f(s), \quad q(s)=\arg\tilde{u}(s(1+i)/\sqrt{2}).$$

R is invertible for *all* $f\neq 0$ (as element in S_α). R takes the form ω_f into ω_f':

$$\omega_f'((\delta_1 p,\delta_1 q),(\delta_2 p,\delta_2 q))=\int ds(\delta_1 p(s)\delta_2 q(s)-\delta_2 p(s)\delta_1 q(s)).$$

Its related Poisson bracket is by definition

$$\{F,G\}_{\omega_f'}=\int ds\left(\frac{\delta F}{\delta p(s)}\frac{\delta G}{\delta q(s)}-\frac{\delta F}{\delta q(s)}\frac{\delta G}{\delta p(s)}\right)$$

and $\{F,G\}_{\omega_f}=\{F\circ R^{-1},G\circ R^{-1}\}_{\omega_f'}\circ R$.

The vector fields (in Sect. 1) X_{4n+2}, $n=0,1,\ldots$, are symplectic with respect to ω_f and therefore the generalized moments are

$$c_n(u)=(-1)^n 2^{-1}(2\pi)^{-1/2}\int dx\,dy[\tilde{f}^{(4n+2)}((x-iy)(1-i)/\sqrt{2})+\tilde{f}^{(4n+2)}((y-ix)(1-i)/\sqrt{2})]u(x)u(y),$$
$$\tilde{f}^{(n)}(z)=(\partial/\partial z)^n\tilde{f}(z), \quad z=x+iy,$$

where $i_X\omega_f=-dc_n$ are the constants of motion for the heat equation. The c_n are in involution. The Hamiltonian function for the heat equation is c_0 (which is of course not to be confused with the nonconserved physical energy).

The Hamiltonian formalism for the Burgers equation is now defined by the symplectic form $\Omega_f=\hat{A}*\omega_f$ (with related Poisson bracket $\{F,G\}_{\Omega f}=\{F\circ\hat{A}^{-1},G\circ\hat{A}^{-1}\}_{\omega_f}\circ\hat{A}$). It follows from the above mentioned properties for the heat equation that the functions $C_n=c_n\circ\hat{A}$, $n=0,1,\ldots$, are constants of motion in involution and that $i_{Y_n}\Omega_f = -dC_n$. To expose the complete integrability for the Burgers equation we introduce the variables $(P(s), Q(s)) = [R(A(u))](s)$. Then $\{P(s), Q(s')\}_{\Omega_f} = \delta(s-s')$, $\{P(s), P(s')\}_{\Omega_f} = 0$, the expression $C_0(u) = -\int ds \times s^2 P(s)$, and the Hamiltonian equation $\dot{F} = \{C_0, F\}_{\Omega_f}$ give

$$\dot{P}(s) = 0 \quad \text{and} \quad \dot{Q}(s) = -s^2.$$

Finally I mention that

$$P(s) = f(s) \sum_{n \geq 0} (1/n!) B_n(u) s^n \quad \text{for} \quad u \in S_{1/2},$$

which explains the normalization of b_n in (10).

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[2]R. K. Bullough and P. J. Caudrey, in *Solitons*, Topics in Current Physics, edited by R. K. Bullough and P. J. Caudrey (Springer, Berlin, 1980), p. 52; A. S. Fokas, J. Math. Phys. 21, 1318 (1980). These authors are apparently assuming supplementary hypotheses on the structure of constants of motion and symplectic forms, such as locality. Further, the spaces of initial conditions are not given. For these reasons their results are not necessarily in contradiction with the results obtained here.

[3]The representation is formally linearizable in the sense of M. Flato, G. Pinczon, and J. Simon, Ann. Sci. Ec. Norm. Sup. 10, 405 (1977).

[4]The form of $\hat{A}$ has independently been found by G. Pinczon, unpublished.

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[6]P. R. Chernoff and J. E. Marsden, *Properties of Infinite Dimensional Hamiltonian Systems*, Lecture Notes in Mathematics, Vol. 425 (Springer, Berlin, 1974).