Kac-Moody Algebra is Hidden Symmetry of Chiral Models

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The infinite parameter Kac-Moody algebra $C[t]\otimes G$, whose elements are loops in G and which is related to the vertex operator for the string model when $G=\mathrm{sl}(2c)$, is identified as the hidden-symmetry algebra of the two-dimensional chiral models. These observations suggest that a Kac-Moody Lie algebra is the hidden symmetry of Yang-Mills fields, a phenomenon which, if true, might lead to complete integrability and nonperturbative information. This algebra, also relevant to integrable soliton theory, may elucidate the classical and quantum inverse method for the chiral theory.

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This paper identifies the infinite parameter Lie algebra responsible for the nonlocal currents[1] in the general two-dimensional chiral models to be the affine Kac-Moody[2-4] subalgebra $C[t]\otimes G$ and indicates how this observation could lead to new information for chiral and gauge fields and its consequence for strong-interaction theory. Here G is the algebra of the chiral-theory symmetry group. In order to clarify the notation for the general audience, observe that a simple representation of the generators of the algebra $C[t]\otimes G$ is $M_e^{(n)}=T_e\otimes t^n$ where t is an indeterminate, $n=1,2,\ldots,\infty$, and T_e are the generators of the finite parameter algebra G. It then follows, given $[T_e, T_f]=c_{efg}T_g$, that $[M_e^{(n)}, M_f^{(m)}] = c_{efg}M_g^{(n+m)}$. This is familiar to physicists as current algebra. $C[t]\otimes G$ is an infinite-dimensional Lie algebra defined over the field $C[t]$, the ring of polynomials in t with values in C, the complex numbers.

In light of the major observation in this field,[5] namely the close analogy of the chiral model to the functional formulation[5-7] of Yang-Mills theory, it is possible that the continuous symmetry group generated by $C[t]\otimes \mathrm{SU}(N)$ is an invariance of the Yang-Mills action *in addition* to the fifteen-parameter conformal group and the SU(N) gauge transformations. In any case, it is remarkable that the elements of $C[t]\otimes G$ may be regarded as loops in G when C is restricted to S^1.[4] Furthermore the exponential generating function for $A_1^{(1)} \equiv (C\otimes G)\otimes C_z$ is known to be similar to the vertex operator in the string model.[3] The algebra $A_1^{(1)}$ is also relevant for supersymmetry.[8] Among other questions that still need to be answered is in what sense does the loop-space variable $\psi[\xi] = Pe^{\oint A\cdot d\xi}$ or the conventional variable $A_\mu{}^a(x)$ carry a representation of the loop algebra $C[t]\otimes\mathrm{SU}(N)$. Surprisingly, the classical theory for G goes over extensively in general to $C[t,t^{-1}]\otimes G$, the algebra defined over the Laurent polynomials. For instance, $C[t,t^{-1}]\otimes G$ possesses a family of irreducible representations analogous to the finite-dimensional representations of G.[3,4] Previously a construction of $C[t,t^{-1}]\otimes \mathrm{sl}(2c)$ was given in the mathematical literature using differential operators.[3] This may prove to be the useful representation for Yang-Mills loop space.

The Kac-Moody algebra has also recently been shown to be relevant in the Bäcklund transformations of soliton systems.[9] Although the existence of conserved commuting charges is characterized by an infinite-parameter Abelian algebra, the non-Abelian Kac-Moody also appears, inherent to the differential operators of the integrability conditions. Application of this algebra to the integrability problem in the chiral fields might help to resolve the unsolved initial-value problem of the classical theory and the proposed simpler problem of the quantum inverse method.[10]

It is emphasized that an explicit construction of the Kac-Moody algebra $C[t]\otimes G$ for a general simple Lie algebra G is given in this paper.

The Lagrangian density for the general class of chiral models is $\mathcal{L}(x)=\frac{1}{16}\operatorname{tr}\partial_\mu g(x)\partial_\mu g^{-1}(x)$. The matrix field $g(x)$ is an element of the group generated by the algebra G. The equations of motion are $\partial_\mu A_\mu=0$ where $A_\mu\equiv g^{-1}\partial_\mu g$. The infinitesimal transformations associated with the infinite set of nonlocal currents are given by

$$\delta^n g = -g\Lambda^n \equiv -g\Lambda_e{}^n\rho_e\,,\quad \Lambda^{n+1}(x,t)=\int_{-\infty}^x dy\, D_0\Lambda^n(y,t)\equiv\int_{-\infty}^x dy\{\partial_0\Lambda^n(y,t)+[A_0(y,t),\Lambda^n(y,t)]\}\,, \qquad (1)$$

$$\Lambda^0 = T\equiv T^e\rho^e \qquad (2a)$$

$$\Lambda^1=[x,T]=[\textstyle\int_{-\infty}^x dy\,A_0(y,t),\,T]\,,\ \text{etc.} \qquad (2b)$$

Here ρ^e is constant and T^e are the matrix generators of G. Let us also define $\delta_{(\rho)}{}^n g = \delta_e{}^n g\rho_e = -g\Lambda_e{}^n \rho_e = \delta_g{}^n$. Equation (1) (i) is a symmetry of the equations of motion [see (3)] and (ii) shifts $\mathcal{L}(x)$ by a total divergence without use of the equations of motion[11] [see (14)]:

$$\partial_\mu[(g^{-1}+\Lambda^n g^{-1})\partial_\mu(g-g\Lambda^n)] = \partial_\mu A_\mu - \partial_\mu D_\mu \Lambda^n + O(\rho^2)\,. \tag{3}$$

With the equations of motion, $\partial_\mu A_\mu = 0$, and $\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0$, when $\partial_\mu D_\mu \Lambda^n = 0$, $D_\mu \Lambda^n = \epsilon_{\mu\nu}\partial_\nu \Lambda^{n+1}$, then $\partial_\mu D_\mu \Lambda^{n+1} = 0$.

The generators $M_e{}^{(n)}$ for the hidden symmetry group are constructed from the infinitesimal transformations.[6] For $\delta^n g \equiv \rho^e \delta_e{}^n g$, we have

$$M_e{}^{(n)} = -\int d^2y\, \delta_e{}^n g(y)\,\delta/\delta g(y). \tag{4}$$

To identify the group, we compute the commutators $[M_e{}^{(m)}, M_d{}^{(n)}]$. From (1), we find explicitly that

$$[M_c{}^{(0)}, M_b{}^{(n)}] = c_{cba} M_a{}^{(n)} \tag{5}$$

$$[M_c{}^{(1)}, M_b{}^{(n)}] = c_{cba} M_a{}^{(n+1)}, \tag{6}$$

where c_{cba} are the structure constants of G: $[T_c, T_b] = c_{cba} T_a$. From (6) and the Jacobi identity it follows for $n, m = 0, 1, 2\ldots$ that

$$[M_c{}^{(m)}, M_b{}^{(n)}] = c_{cba} M_a{}^{(m+n)}\,. \tag{7}$$

This is the infinite parameter algebra $\mathbb{C}[t]\otimes G$.

The proof (by induction) of (5), (6), and (7) is as follows: From (1) and (4), to order $\rho_c\sigma_b$,

$$[M_c{}^{(m)}, M_b{}^{(n)}]\rho_c\sigma_b$$

$$= \int d^2x \{\delta_{(\sigma)}{}^n[g(x) + \delta_{(\rho)}{}^m g(x)] - \delta_{(\sigma)}{}^n g(x) - \delta_{(\rho)}{}^m[g(x) + \delta_{(\sigma)}{}^n g(x)] + \delta_{(\rho)}{}^m g(x)\}\,\delta/\delta g(x) \tag{8a}$$

$$= -\int d^2x\, g\{\sigma_b[\Lambda_b(g - g\rho_c\Lambda_c) - \Lambda_b] - \rho_c[\Lambda_c{}^m(g - g\sigma_b\Lambda_b{}^n)\Lambda_c{}^m] - \sigma_b\rho_c[\Lambda_c{}^m, \Lambda_b{}^n]\}\,\delta/\delta g. \tag{8b}$$

Assume $[M_c{}^{(1)}, M_b{}^{(n)}] = c_{cba} M_a{}^{n+1}$ for $n = N$. Then

$$\Lambda_b{}^N(g - g\rho_c\Lambda_c{}^1) - \Lambda_b{}^N = \rho_c\{[\Lambda_c{}^1, \Lambda_b{}^N] - [\Lambda_b{}^{N+1}, T_c] + c_{bca}\Lambda_a{}^{N+1}\}\,. \tag{9}$$

Then, using (2a), (9), and $\chi(g - g\rho_c\Lambda_c{}^n) = \chi - \rho_c\Lambda_c{}^{n+1} + O(\rho^2)$ and $\partial_y\chi = A_0$, we have

$$\sigma_b\rho_c[M_c{}^{(1)}, M_b{}^{(N+1)}] = -\int d^2x\, g(x)\{\sigma_b\int_{-\infty}^x dy\,(D_0[\Lambda_b{}^N(g - g\rho_c\Lambda_c{}^1) - \Lambda_b{}^N] - [D_0\rho_c\Lambda_c{}^1, \Lambda_b{}^N])$$
$$+\sigma_b\rho_c([\Lambda_b{}^{N+2}(x), T_c] - [\Lambda_c{}^1(x), \Lambda_b{}^{N+1}(x)])\}\,\delta/\delta g(x)$$
$$= -\int d^2x\{\nabla_b\rho_c c_{bca}\int_{-\infty}^x dy\, D_0\Lambda_a{}^{N+1}\}\,\delta/\delta g(x)\,, \tag{10}$$

or equivalently

$$[M_c{}^{(1)}, M_b{}^{(N+1)}] = c_{cba} M_a{}^{N+2}\,. \tag{11}$$

Also, $[M_c{}^{(1)}, M_b{}^{(1)}] = c_{cba} M_a{}^{(2)}$. Note that equation (9) is true for $N = 1$ because $\Lambda_b{}^1(g - g\rho_c\Lambda_c{}^1) - \Lambda_b{}^1 = -\rho_c[\Lambda_c{}^2, T_b]$ and $\Lambda_c{}^2 = [\chi^2, T_c] + \frac{1}{2}[\chi, [\chi, T_c]]$, where $\chi^2(x,t) = \int_{-\infty}^x dy\{\partial_0\chi(y,t) + \frac{1}{2}[A_0(y,t), \chi(y,t)]\}$. Therefore, from (11) we have proved (6). The proof of (5) is similar, where

$$\Lambda_b{}^N(g - g\rho_c T_c) - \Lambda_b{}^N = \rho_c\{-[\Lambda_b{}^N, T_c] + c_{bca}\Lambda_a{}^N\}\,.$$

To prove (7), assume for $n = N$ and $m = M$,

$$[M_c{}^{(n)}, M_b{}^{(m)}] = c_{cba} M_a{}^{(n+m)}\,. \tag{12}$$

Equation (12) is true for $m = 1$ and all n. Therefore $[M_c{}^{(n+1)}, M_b{}^{(m)}] = c_{cba}{}^{(n+m+z)}$ for $m = 1$ and all n. Now we prove, using the Jacobi identity, that given (12), $[M_a{}^{(N)}, M_b{}^{(M+1)}] = c_{abc} M_c{}^{(M+N+1)}$:

$$c_{cdb}[M_a{}^{(N)}, M_b{}^{(m+1)}] = [M_a{}^{(N)}, [M_c{}^{(m)}, M_d{}^{(1)}]] = [M_c{}^M, [M_a{}^N, M_d{}^1]] + [M_d{}^1, [M_c{}^M, M_a{}^N]]$$
$$= (c_{ade}c_{ceg} + c_{cae}c_{deg})M_g{}^{M+N+1} = c_{cdb}c_{abg}M^{M+N+1}. \tag{13}$$

That (1) shifts $\mathcal{L}(x)$ by a total divergence is now proved by induction. The transformation (1) shifts the Lagrangian density by $\rho_e \Delta_e{}^n \mathcal{L}(x) = -\rho_e[M_e{}^{(n)}, \mathcal{L}(x)] = \frac{1}{8}\,\mathrm{tr}\, A_\mu \partial_\mu \Lambda^n$. Therefore, to first order in $\rho_e \sigma_d$, from (6) and Jacobi's identity, we have

$$\rho_e \sigma_d c_{eda} \Delta_a{}^{n+1}\mathcal{L}(x) = -c_{ead}[M_a{}^{(n+1)}, \mathcal{L}(x)]\rho_e \sigma_d = -[[M_e{}^{(n)}, M_d{}^{(1)}], \mathcal{L}(x)]\rho_e \sigma_d$$
$$= \{[M_e{}^{(n)}, \Delta_d{}^1\mathcal{L}(x)] - [M_d{}^{(1)}, \Delta_e{}^n \mathcal{L}(x)]\}\rho_e \sigma_d. \tag{14}$$

Since

$$\Delta_d{}^1\mathcal{L}(x) = \tfrac{1}{8}\partial_\mu \epsilon_{\mu\nu}\,\mathrm{tr}(A_\nu + \tfrac{1}{2}[\partial_\nu \chi, \chi])T_d,$$

then

$$[M_e{}^{(n)}, \Delta_d{}^1\mathcal{L}(x)] = \tfrac{1}{8}\partial_\mu \epsilon_{\mu\nu}\,\mathrm{tr}([A_\nu, \Lambda_e{}^n] + [\partial_\nu \chi, \Lambda_e{}^{n+1}])T_d.$$

We assume that $\Delta_e{}^n \mathcal{L}(x)$ can be written as a total divergence: $\Delta_e{}^n \mathcal{L}(x) = \partial_\mu \kappa_\mu{}^{ne}([g], x)$. (This is true for $n=1$.) Then

$$[M_d{}^1, \Delta_e{}^n \mathcal{L}(x)] = -\partial_\mu \kappa_\mu{}^{ne}([g - g\Lambda_d{}^1], x) + \partial_\mu \kappa_\mu{}^{ne}([g], \chi)$$

is also a total divergence and $\rho_e \sigma_d c_{eda} \Delta_a{}^{n+1}\mathcal{L}(x)$ is a total divergence.

In conclusion, Eqs. (1), (2), and (7) are an explicit construction of the Kac-Moody algebra $\mathbb{C}[t]\otimes G$. Since the infinitesimal transformations (1) form an algebra, they generate finite field transformations which form the hidden symmetry group.

All these symmetry considerations have been carried out in the classical formulation of the chiral theory, that is to say the infinitesimal field transformations rather than the quantum brackets have carried the representation of the group. This has been done for two reasons. First, for $G = O(3)$ the nonlinear sigma model, a specific constraint requires quantization via Dirac brackets. (The hidden symmetry is responsible for factorization of the two-particle S matrix and no particle production and thus remains in the quantum theory.) Since constraints differ for various choices of G, it is impossible to discuss quantum brackets for G in a general way: To quantize, we must specify the theory. Secondly, the motivation for this paper comes from the need to develop a systematic nonperturbative approximation for the Yang-Mills theory. The chiral models have so far been relevant via a classical connection to the functional Yang-Mills fields. And it is in the effort to maintain a general matrix $g(x)$ which is then replaced by the matrix $\psi[\xi] = Pe^{\oint A\cdot d\xi}$ in the study of symmetry which led to the emphasis on classical notation.

The next step in this work is to see whether or not this algebra exists in the gauge theory. If it does, I propose the following consequences: (1) In accordance with Ref. 5, the symmetry will lead us to complete integrability of Yang-Mills theory, either in loop space or in the conventional formulation. (2) If the symmetry survives quantization, it will imply nontrivial restrictions on the fundamental integral in the nonperturbative region, which we can calculate in a systematic way. It would thus provide a controlled approximation to strong interaction theory.

I am heartened that the symmetry algebra uncovered in the chiral models shares the properties we believe to be useful for a nonperturbative picture of the strong interactions, namely that the fundamental variable $\psi[\xi]$ is defined in loop space and that $\mathrm{tr}\psi$ is related to the string wave functional.

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Higher-Order Effects in Wide-Angle Bremsstrahlung

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The origin of the large higher-order effects in radiative corrections is traced to charged-lepton scattering. The long radiative tail of the one-photon inclusive cross section, which does not arise in lowest order, is responsible for these large effects.

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Nonperturbative methods for radiative corrections, initiated in the basic work of Yennie, Frautschi, and Surra,[1] have been further developed in the important paper of Grammer and Yennie.[2] On the other hand, the more rigorous aspects of the nonperturbative treatment of infrared divergences have been vigorously studied by Zwanziger.[3] Recently[4,5] I have introduced rigorous and practical nonperturbative methods to effect radiative corrections. Surprisingly, these methods predicted[5] that in the recent muon scattering experiments[6-8] the elastic contribution is much larger, in the very inelastic regime, than the conventional one-emitted-photon (1γ) bremsstrahlung cross section.[9] The choice of a technique to carry out the radiative corrections affects substantially the extracted nonradiative structure functions and in particular the scaling violations for small x. It is thus important to understand and possibly check by experiment which approach is correct.

It is shown in this paper that these large higher-order effects[5] originate from a long radiative tail to the *hard bremsstrahlung cross section*, not present if only one-photon emission is taken into account.[10] In effect, in the very inelastic regime, there is a large probability that the hard photon is accompanied by collinear radiation. I compute this "radiatively corrected" hard bremsstrahlung cross section as a function of the photon energy and compare it to the conventional Bethe and Heitler[11] formula. The two results may be submitted to experimental test.

In the one-photon exchange approximation the bremsstrahlung differential cross section (DCS) in charged lepton-proton scattering, including the emission of an arbitrary number of collinear unobserved photons from the lepton vertex and the corresponding infrared virtual corrections, is given by

$$\frac{\omega\, d\sigma}{d\Omega'\, dE'\, d\omega} = \frac{\alpha^3}{(2\pi)^2}\frac{p'}{p}\int \frac{d^4q}{(q^2)^2}\, d\Omega_k\, W_j^{\,\mathrm{NR}}(P,q)\, T_j(p,p';k)\hat{E}(p,p';p-q-p'-k). \tag{1}$$

Here, $P(M,0)$, $p(E,\vec{p})$, $p'(E',\vec{p}\,')$, $k(\omega,\vec{k})$, and $q(q^0,\vec{q})$ are the momenta of the proton, the incident and scattered lepton, and the observed and exchanged photon, respectively; the vectors $\vec{p}\,'$ and $\vec{k}$ point in the solid angles Ω' and Ω_k. $W_j^{\,\mathrm{NR}}$ denotes the usual nonradiative proton structure functions ($j=1,2$),