Anomalous Fermion Production by a Julia-Zee Dyon

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(Received 17 August 1981)

The emission of light charged fermions from a one-dimensional potential step and from a Julia-Zee dyon is analyzed. Even for the case of massless fermions, this emission process violates chirality and provides a simple illustration of the axial anomaly.

PACS numbers: 11.30.Rd, 11.10.Np, 14.80.Hv, 14.80.Kx

Since its discovery[1] in 1969, the axial anomaly has had a number of interesting and important applications. Perhaps most dramatic are those which result in the actual production of fermions with quantum numbers inconsistent with classical conservation laws or quantum perturbation theory.[2] We will show that such anomalous fermion production occurs in the background of a Julia-Zee dyon.[3] Let us first examine a simpler case.

(i) *One-dimensional electrostatic step.*—Consider a charged fermion moving in the one-dimensional potential $A_0(x)$ shown in Fig. 1 according to the Dirac Hamiltonian

$$H = -i\alpha \cdot \partial_x + \beta m + eA_0. \tag{1}$$

When $2m < v$ it is energetically favorable to produce particle-antiparticle pairs in the electric field near the origin, sending the particle with positive charge off to the left and the antiparticle to the right. In the limit $m = 0$ (which we will henceforth assume) this emission should be least suppressed. However, it explicitly violates the conservation of the chiral charge

$$Q^5 = \int dx\, \psi^\dagger \gamma_5 \psi = (N_l - \bar{N}_l) - (N_r - \bar{N}_r), \tag{2}$$

where $\gamma^5 = -\alpha$, $N_l, N_r, \bar{N}_l, \bar{N}_r$ are the number of left- or right-moving particles or antiparticles, respectively. For the fermion emission described above $\Delta Q^5 = 2$! The quantity Q^5 is conserved on the "classical" or first-quantized level. However, the corresponding current contains an anomaly[4] in the second-quantized field theory:

$$\partial_\mu j^{\mu,5} = \partial_\mu \bar{\psi} \gamma^\mu \gamma^5 \psi = (e/2\pi) F^{\mu\nu} \epsilon_{\mu\nu}. \tag{3}$$

In the large-time limit one expects a steady-state flux of the chirality Q^5,

$$j^{x,5}(x)\Big|_{-L}^{L} = \int_{-L}^{L} dx\, \partial_\mu j^{\mu,5} = (e/\pi)\int_{-L}^{L} dx\, \partial_x A_0 = v/\pi, \tag{4}$$

for L large. Let us compute the flux of chirality directly and compare with this prediction of the anomaly.

We will choose as the initial state $|i\rangle$ of the fermionic system one which looks like the vacuum for large $|x|$. In the language of second quantization, we expect that for $x \gg l$ those states with $E > \frac{1}{2}v$ should be empty ("positive"-energy states if the constant potential $\frac{1}{2}v$ is ignored) while those with $E < \frac{1}{2}v$ should be filled. Similarly for large negative x, states with $E > -\frac{1}{2}v$ should be empty and those with $E < -\frac{1}{2}v$ filled. Such a state can be defined by expanding the Heisenberg field operator $\psi(x)$ in an appropriate basis:

$$\psi(x) = \sum_{\kappa=\pm\frac{1}{2}} \int_0^\infty d\epsilon [a_\kappa{}^\epsilon \varphi_\kappa{}^{\epsilon-\kappa v}(x) + b_\kappa{}^{\epsilon\,\dagger} \varphi_\kappa{}^{-\epsilon+\kappa v}(x)]. \tag{5}$$

Here the basis functions $\varphi_\kappa{}^E$ are chosen to be a complete orthonormal set (normalized to δ functions in energy) which vanish for $\kappa x \gg l$ and satisfy $H\psi = E\psi$ for $-\kappa x \gg l$. With such a choice of $\varphi_\kappa{}^E$ the initial state defined by

$$a_\kappa{}^\epsilon |i\rangle = b_\kappa{}^\epsilon |i\rangle = 0 \tag{6}$$

satisfies the requirements outlined above. A simple method for obtaining such a basis uses the Hamiltonian (1) but treats the regions $x > 0$ and $x < 0$ as completely independent. Hermiticity in each region is maintained if we only consider functions whose upper and lower Dirac compo-

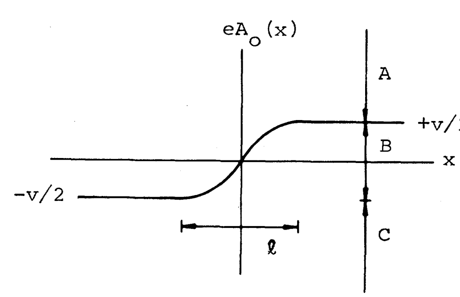

FIG. 1. The one-dimensional electrostatic potential energy eA_0.

nents obey $\varphi_{\kappa,u}(0)/\varphi_{\kappa,l}(0)=\tan\delta_\kappa$; thus, for $\alpha=\tau_2$,

$$\varphi_\kappa^{E}(x)=\begin{cases}0, & \kappa x>0\\ \dfrac{1}{(2\pi)^{1/2}}\begin{pmatrix}\sin[-\int_0^x eA_0(x)+Ex+\delta_\kappa]\\ \cos[-\int_0^x eA_0(x)+Ex+\delta_\kappa]\end{pmatrix}, & \kappa x<0.\end{cases} \tag{7}$$

Note that the state (7) necessarily contains a mixture of chiralities. Our physical requirements on the initial state $|i\rangle$ have prevented the conventional choice that the $\varphi_\kappa^E(x)$ be eigenstates of the Hamiltonian (1). Although such states can be chosen to have definite chiralities they will unavoidably mix positive and negative energies. A left-moving eigenstate with energy E lying in region B (Fig. 1) will be interpreted as a negative-energy state on the right, but must travel without reflection to the left where it is viewed as a positive-energy state.

It is now a straightforward matter to determine the flux of particles and antiparticles present in the large-time limit of $|i\rangle$. As we will see, the only contribution comes from those states with energy in region B, initially confined to the right-hand side, $\varphi_{-1/2}{}^E$. Their large-time limit is as follows: (a) The left-moving component travels through the step and for large time approaches a left-moving eigenstate of (1). (b) The right-moving component travels off to the right and disappears; there is no incoming or reflected wave from the left to replenish it. As a result of (a) the left-moving states with energy E between $-\frac{1}{2}v$ and $+\frac{1}{2}v$ appear filled at large times. Since each carries a probability flux of $1/2\pi$, the integrated flux of chirality carried off to the left by these particles if $\mathfrak{F}_+=v/2\pi$.

Similarly because of (b) the large-time limit of $|i\rangle$ differs on the right from the vacuum because the right-moving states in region B are empty. If these empty states are interpreted as right-moving antiparticles we get a further outgoing chiral flux $\mathfrak{F}_-=\mathfrak{F}_+$. Since for regions A and C the states on the right and left are either both filled or both empty, that condition remains true in the large-time limit and no further flux results. Thus the total chiral flux is $\mathfrak{F}_+ + \mathfrak{F}_- = v/\pi$ in precise agreement with the prediction (4) of the axial anomaly. We now turn to the three-dimensional case which is remarkably similar to the situation above.

(ii) *Three-dimensional Julia-Zee dyon.*—In the simplest static gauge the vector potential corresponding to a Julia-Zee dyon has the form[3]

$$A_i^{\,a}=\epsilon^{aji}(\hat{r})^j[K(r)-1]\,\frac{1}{gr},\qquad A_0^{\,a}=-(\hat{r})^a J(r)\,\frac{1}{gr}. \tag{8}$$

Here i and 0 indicate space and time directions, a is an SU(2) vector index, g is the Yang-Mills coupling constant, $r=|\vec{r}|$, and $\hat{r}=\vec{r}/r$. For large r $K(r)\sim O(e^{-\mu r})$ and $J(r)/r\sim -v+Q/r+O(e^{-\mu r})$. We will consider the case where the constants v and Q are positive. Asymptotically, the field configuration specified by (8) can be gauge transformed to $\tau_3\mathfrak{A}_\mu$ where $\mathfrak{A}_\mu$ is the vector potential of an Abelian magnetic monopole with pole strength $1/g$ and electric charge Q/g.

If a massless, SU(2) doublet, Fermi field $\psi(\vec{r},t)$ is coupled to the dyon, the corresponding Hamiltonian is

$$H=\vec{\alpha}\cdot(-i\nabla-\tfrac{1}{2}g\vec{A}^a\tau^a)+\tfrac{1}{2}gA_0^{\,a}\tau^a. \tag{9}$$

For large r this Hamiltonian simplifies and its eigenfunctions $\varphi^E(\vec{r})$ have the form

$$\varphi^E(\vec{r})=U(\vec{r})\begin{pmatrix}\varphi^{E,+}(\vec{r})\\ \varphi^{E,-}(\vec{r})\end{pmatrix}, \tag{10}$$

where $U(\vec{r})$ is the gauge transformation referred to above and $\varphi^{E,\pm}$ are four-component spinors which are eigenstates of the Abelian Dirac Hamiltonian for a fermion of charge $\pm\frac{1}{2}g$ moving in the vector potential $\mathfrak{A}_\mu$. Again because $\mathfrak{A}_0$ is nonzero at spatial infinity, one expects that this fermionic system should be unstable. The presence of fermion fields φ^+ and φ^- with opposite charges can be exploited by creating and moving to infinity a φ^+ fermion with charge $+\frac{1}{2}g$ and a φ^- antifermion also with charge $+\frac{1}{2}g$. If the fermions had a mass m, this would require a rest mass energy $2m$ but liberate an electrostatic energy $2(v/2)$. Hence we expect instability against fermion emission when $2m<v$ [just as in (i)].

The specification of the initial fermionic state $|i\rangle$ in the background of the dyon and its large-time limit follows exactly the same procedure as the simpler example (i) above. Again (see Fig. 2) the state $|i\rangle$ must have all states with charge $\pm\frac{1}{2}g$ and $E > \mp\frac{1}{2}v$ empty while states with charge $\pm\frac{1}{2}g$ and $E < \mp\frac{1}{2}v$ must be filled. This is accomplished if we replace the four-component functions $\varphi_{\pm 1/2}{}^E$ in the expansion (5) by eight-component functions $\varphi_{\alpha,\pm 1/2}{}^E$ which obey similar requirements: For large r, $\varphi_{\alpha,\kappa}{}^E$ is an eigenstate of the Hamiltonian (9) with eigenvalue E (the extra quantum numbers α are chosen below) and after gauge transformation, as in Eq. (10), contains only an upper ($I_z = \kappa = +\frac{1}{2}$) or a lower ($I_z = \kappa = -\frac{1}{2}$) isotopic component. Just as in the one-dimensional case, these basis functions cannot be eigenfunctions of the true Hamiltonian since that operator mixes these two asymptotic behaviors.

The flux of chirality $Q^5 = \int d^3x\, \psi^\dagger\gamma^5\psi$ carried by the emitted fermions present in the long-time limit of $|i\rangle$ can be computed quite simply if the $\varphi_{\alpha,\kappa}{}^E$ are chosen to be eigenstates of $\vec{J}^2$ and J_z following the analysis of Kazama, Yang, and Goldhaber.[5] Here $\vec{J} = \vec{L} + \frac{1}{2}\vec{\sigma}$ is the total angular momentum with $\vec{L} = \vec{r} \times (-i\nabla - g\kappa\vec{\mathfrak{A}}) - \kappa\hat{r}$. The new index α represents the triple j, m, i. The quantity $j(j+1)$ is the eigenvalue of $\vec{J}^2$ while m is the eigenvalue of J_z. For $j > 0$ there are two independent solutions with fixed charge and the above eigenvalues. These are distinguished by $i = 1, 2$ and can be chosen to be orthogonal and to obey

$$\vec{\sigma}\cdot\hat{r}\,\varphi_{j,m,1,\kappa}{}^E = -\varphi_{j,m,2,\kappa}{}^E. \tag{11}$$

For $j = 0$, there is only one solution, labeled $i = 3$, which is an eigenstate of $\vec{\sigma}\cdot\hat{r}$ with eigenvalue 2κ. Because of Eq. (11) and the orthonormality of the states with $i = 1$ and 2, there is no net chirality carried off by particles in these states: e.g., $\int d\Omega_{\hat{r}}\, \hat{r}\cdot\bar{\varphi}_1\vec{\gamma}\gamma^5\varphi_1 = 0$, while a similar calculation shows that $\varphi_{0,0,3;\kappa}{}^E$, being an eigenstate of $\vec{\sigma}\cdot\hat{r}$, carries a maximum chiral flux κ/π in a manner identical to the chiral flux of $+1/2\pi$ carried by both left- and right-moving fermions in our one-dimensional example.

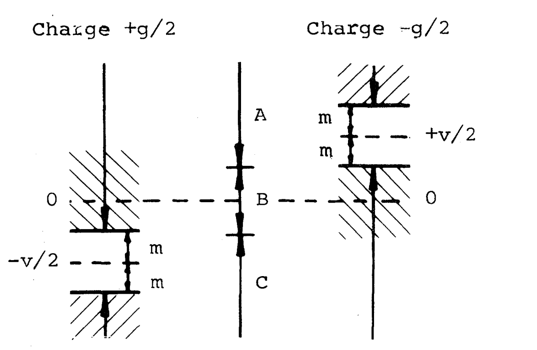

FIG. 2. Energy spectrum of particles with charge $\pm\frac{1}{2}g$ ($I_z = \pm\frac{1}{2}$) and mass m far from a Julia-Zee dyon.

Thus the flux of chirality radiated by the dyon is carried by fermions in this special $j = 0$, $i = 3$ state. Just as before we begin with those states with $I_z = -\frac{1}{2}$ and energy in region B (Fig. 2) filled, while the corresponding states with $I_z = +\frac{1}{2}$ are empty. As time evolves, the incoming part of the $I_z = -\frac{1}{2}$ state is reflected by the dyon. Because of chirality conservation the reflected outgoing wave must carry a flux of chirality equal and opposite to that of the incoming wave. This means, as we saw above, that it must have opposite charge,[6] or $I_z = +\frac{1}{2}$. Hence in the limit of large time the $I_z = +\frac{1}{2}$, $j = 0$ outgoing states with energy in the region B are occupied. These can be interpreted as radiated particles carrying a flux of chirality $\mathfrak{F}_+ = v/2\pi$. Similarly the outgoing parts of the filled $I_z = -\frac{1}{2}$ states disappear off to infinity as the time evolves and are not replenished by scattering from the incoming $I_z = +\frac{1}{2}$ states which are empty. These empty outgoing states (with a chirality flux $+1/2\pi$ when viewed as outgoing antiparticles) carry a net chiral flux $\mathfrak{F}_- = \mathfrak{F}_+$. The resulting total flux of chiral charge is $\mathfrak{F}_+ + \mathfrak{F}_- = v/\pi$. This can easily be compared with the prediction of the axial anomaly,

$$\partial_\mu\bar{\psi}\gamma^\mu\gamma^5\psi = (g^2/32\pi^2)F^{\mu\nu,i}F^{\rho\sigma,i}\epsilon_{\mu\nu\rho\sigma}. \tag{12}$$

As before, the steady-state chiral flux is given by the spatial integral of either side of Eq. (12). For the case where $F_{\mu\nu}{}^a$ is the field strength of a Julia-Zee dyon we have[7]

$$\begin{aligned}(g^2/16\pi^2)\int d^3x\, F^{\mu\nu,a}\tilde{F}_{\mu\nu}{}^a &= -(g^2/4\pi^2)\int d^3x\, \partial_i A_0{}^a B_i{}^a \\ &= \lim_{R\to\infty}\left[-(g^2/4\pi^2)\int d\Omega_{\hat{r}}\, A_0{}^a(\hat{r}R)B_i{}^a(\hat{r}R)\hat{r}^i R^2\right] \\ &= v/\pi \end{aligned} \tag{13}$$

in precise agreement with our above result.

We conclude that the emission of massless fermions by a Julia-Zee dyon provides a relatively simple situation where the physics of the axial anomaly can be seen quite directly. The flux of chirality carried off by the emitted fermions implies an actual creation of chiral charge only if the intrinsically ambiguous vacuum matrix element of the chiral current does not give a compensating flux. With a gauge-invariant definition of the current, the vacuum far from the dyon is not affected by the asymptotic limit of A_0 and this vacuum matrix element vanishes.

One of the authors (N.H.C.) thanks S. Coleman, G. Gibbons, N. Parsons, and F. Wilczek for interesting discussions. This research was supported in part by the U. S. Department of Energy.

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Two-Cluster Channel Decoupling of the Alt-Grassberger-Sandhas *N*-Body Equations

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(Received 9 September 1981)

As an extension of a four-body investigation by Haberzettl and Sandhas, a partial decoupling of the N-body equations by Alt, Grassberger, and Sandhas is performed in such a way that only transitions into a given type of two-cluster configuration remain coupled. Taking into account particle identity, one then has to solve only a single-channel integral equation. All other reaction channels are obtained by quadratures. The resulting expressions resemble optical potential models.

PACS numbers: 11.80.Jy, 24.10.Dp

Up to now numerical investigations making use of Faddeev-type N-body equations have been performed to my knowledge only for systems consisting of three or four bodies. Limited computer capacities on the one hand and the high degree of coupling of existing N-body equations on the other hand are the primary reasons for this rather meagre practical turnout of in principle exact theories. In the present paper, I will show that it is possible for at least one of the N-body theories, namely the one proposed by Alt, Grassberger, and Sandhas (AGS),[1] to remove the high degree of coupling and thus make the N-body problem more amenable to numerical and also certain theoretical (see concluding remarks) investigations.

In what follows I will only consider transitions between arrangements of the N particles into two clusters. There are $2^{N-1}-1$ such two-cluster configurations. Except for one remark below, I will not deal here with breakup explicitly. In the AGS theory, as in many other N-body theories, one has to treat simultaneously in a coupled system of $2^{N-1}-1$ integral equations all transitions between one particular incident two-cluster configuration and all $2^{N-1}-1$ possible final two-cluster channels, even if one is interested in only one of these transitions. (For more details, see Ref. 2.) What I want to show here is that a considerable amount of this coupling can be removed in a quite straightforward manner. While it would be possible, in principle, to decouple the equations completely (i.e., have one integral equation for each transition), I do not want to do so, because this would entail symmetrization problems when dealing with identical particles. I rather want to perform the decoupling in such a manner that only transitions into final channels belonging to the same type of two-cluster configurations remain coupled. Under "type of two-cluster configurations" I subsume all channels that have the same number of particles in the smaller of the two clusters. Hence, denoting this number by n, the degree of coupling I want to achieve is $\binom{N}{n}$, which is smallest for $n=1$, namely just N. (For the special case $N=4$ and $n=1$, this decoupling was