Nonuniversal and Anomalous Decay of Boundary Spin Correlations in Inhomogeneous Ising Systems

H. J. Hilhorst and J. M. J. van Leeuwen
Laboratorium voor Technische Natuurkunde, 2600-GA Delft, The Netherlands
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A triangular, half-infinite, nearest-neighbor Ising system is studied whose coupling strengths close to the boundary are weaker than those in the bulk. The deviations decay as $-A/m^y$ with the distance m from the boundary. By means of a repeated star-triangle transformation the boundary spin-spin correlation $g(r)$ is studied for critical bulk coupling. For $y=1$ the behavior is $g(r)\sim 1/r^\eta$ as $r\to\infty$, with a nonuniversal exponent η depending on A. For $0<y<1$ anomalous decay $g(r)\sim\exp[-(r/\xi)^{1-y}]$ is found. Expressions for η and ξ in terms of A and y are given.

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We consider a half-infinite triangular lattice of Ising spins $\frac{1}{2}$ (see Fig. 1) with ferromagnetic nearest-neighbor interactions. The couplings may differ from column to column and are denoted by $K_1(m)$ and $K_2(m)$, with $m=\frac{1}{2},\frac{3}{2},\ldots$ and $m=1,2,\ldots$, respectively. We are interested in the correlation $g(r)$ between two spins at distance r on the boundary of the system. McCoy and Wu[1] showed that for a *homogeneous* critical system, $g(r)$ decays as[2]

$$g(r)\sim 1/r^\eta\,(r\to\infty),\quad \eta=1. \tag{1}$$

They also studied the effects of a boundary magnetic field[1] and of random couplings[3] on $g(r)$. McCoy and Perk[4] studied bulk lattices with a single column of deviating interactions.

Here we calculate for the first time, and by a new method, boundary effects due to deviating interactions in an arbitrary number of columns. We study in particular the case

$$K_i(m)=K_{iB}-A_i/m^y,\quad i=1,2, \tag{2}$$

where the bulk couplings K_{iB} satisfy the criticality condition[5] $2\sinh 2K_{1B}\sinh 2K_{2B}+\sinh^2 2K_{2B}=1$, and where $A_1, A_2>0$. Equation (2) describes, e.g., samples with a nonuniform temperature distribution. Later calculations simplify if we assume a definite ratio between the A_i by putting

$$A_1=\tfrac{1}{2}A\sinh 2K_{1B},\quad A_2=\tfrac{1}{4}A\cosh 2K_{2B}. \tag{3}$$

For $K_{1B}=K_{2B}$ we have $A_1=A_2$. Our results are the following: For $y>1$ the asymptotic behavior (1) remains unmodified. For $y=1$ the correlation function $g(r)$ decays as a power with a *nonuniversal* exponent η given by

$$\eta=1+A/\sinh 2K_{2B}. \tag{4}$$

For $0<y<1$ the correlation exhibits the *anomalous* decay

$$g(r)\sim\exp[-(r/\xi)^{1-y}], \tag{5}$$

where

$$\xi=\left(\frac{1-y}{2A}\right)^{1/(1-y)}\left[\frac{\Gamma(1/2y)\sinh 2K_{2B}}{\pi^{1/2}\Gamma((1+y)/2y)}\right]^{y/(1-y)}. \tag{6}$$

As $y\to 0$, Eq. (5) reproduces the exponential decay common to homogeneous systems above criticality, and $\xi=1/2A$ in agreement with Ref. 1.

Our method of calculation consists in successively transforming the initially given Hamiltonian H_0 into new ones $H_1, H_2,\ldots$, which all refer to a triangular lattice as in Fig. 1, but with dif-

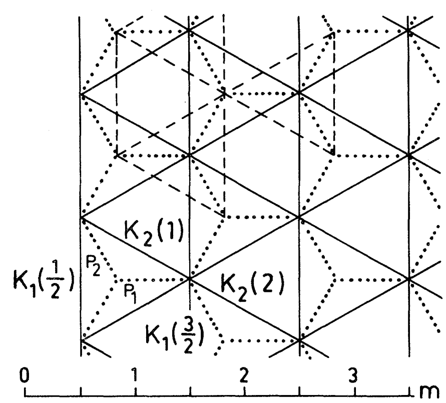

FIG. 1. The initial triangular lattice (solid lines). Nearest neighbors in one column (in neighboring columns) of spins are coupled by the $K_1(m)$ [by the $K_2(m)$]. Each bond K_i depends only on the distance m of its center to the line $m=0$ (see scale). The dotted lines show the equivalent hexagonal lattice with couplings p_1 and p_2 [their labeling, omitted from the figure, is defined by Eq. (7a)]. This hexagonal lattice is again equivalent to a new triangular lattice, of which a few triangles are shown (dashed lines).

ferent sets of couplings, viz. $\{K_i(m,1)\}$, $\{K_i(m,2)\},\ldots$ [by definition $K_i(m,0)=K_i(m)$]. The mapping is based on the star-triangle transformation,[6] which replaces any "triangle" of bonds K_1, K_2, and K_3 by a "star" of bonds p_1, p_2, and p_3 or vice versa. Here $p_i=f_i(K_1,K_2,K_3)$ is implicitly given[7] by

$$K_i=\tfrac{1}{4}\log\left[\frac{\cosh(p_i+p_j+p_k)\cosh(-p_i+p_j+p_k)}{\cosh(p_i-p_j+p_k)\cosh(p_i+p_j-p_k)}\right],$$

with i,j,k cyclic. To obtain H_{n+1} from H_n we first replace each rightward pointing triangle in the nth triangular lattice by a star (see Fig. 1). This yields an intermediate hexagonal lattice. We next convert this into the $(n+1)$th triangular lattice by replacing every leftward pointing star by a triangle. In this second step at the boundary a degenerate case occurs of stars that miss one leg: These are simply converted into single vertical bonds. The column coordinate m is assigned to the bonds of the nth lattice in the same way as in the original lattice. Using this labeling and employing the symmetry of the problem to write $f_i(K_1,K_2,K_2)=f_i(K_1,K_2)$ we have from the first step

$$p_1(m,n)=f_1(K_1(m-\tfrac{1}{2},n),K_2(m,n)),$$
$$p_2(m-\tfrac{1}{2},n)=f_2(K_1(m-\tfrac{1}{2},n),K_2(m,n)), \qquad (7a)$$

and from the second step

$$p_1(m,n)=f_1(K_1(m+\tfrac{1}{2},n+1),K_2(m,n+1)),$$
$$p_2(m+\tfrac{1}{2},n) =f_2(K_1(m+\tfrac{1}{2},n+1),K_2(m,n+1)), \qquad (7b)$$
$$\tanh^2 2p_2(\tfrac{1}{2},n)=1-\exp[-4K_1(\tfrac{1}{2},n+1)],$$

where $m=1,2,\ldots$. The transformation $\{K_i(m,n)\}\to\{K_i(m,n+1)\}$ results explicitly if we eliminate the p_i from Eqs. (7). It fully defines the trajectory in Hamiltonian space. The equations are easily iterated on a computer,[7] but our purpose here is to study them analytically. We remark that even if the initial system was homogeneous, inhomogeneity will penetrate into it from the boundary as one iterates the transformation.

Under this transformation averages of functions of boundary spins satisfy simple recursion relations. Let $g(r,n)$ be the correlation between two boundary spins at distance r in the nth triangular lattice. Then one easily derives

$$g(r,n)=\tfrac{1}{4}\{1-\exp[4K_1(\tfrac{1}{2},n)]\}[g(r-1,n+1)+2g(r,n+1)+g(r+1,n+1)] \qquad (8)$$

for $r=1,2,\ldots$. With the conditions $g(0,n)=1$ and $g(r,0)=g(r)$ we find[8] from Eq. (8)

$$g(r)=\sum_{n=1}^{\infty}2^{-2n}\frac{r}{n}\binom{2n}{n+r}f(n), \qquad (9)$$

$$f(n)=\prod_{j=1}^{n}\{1-\exp[-4K_1(\tfrac{1}{2},j)]\}. \qquad (10)$$

Equations (9) and (10) show that $g(r)$ is known if we can solve $K_1(\tfrac{1}{2},n)$ from Eqs. (7). The $r\to\infty$ behavior of $g(r)$ is, in particular, determined by the $n\to\infty$ behavior of $K_1(\tfrac{1}{2},n)$, and can be obtained from it via a steepest-descent calculation of the sum in Eq. (9).

Our analytic treatment of Eqs. (7) is inspired by a computer calculation. Starting from a homogeneous system we observed that after a small number of iterations of the transformation the $K_i(m,n)$ become slowly varying in m and n. Moreover they appear to tend to a similarity solution $\tilde{K}_i$ depending only on m/n. We decide therefore to look for smooth solutions of Eqs. (7) so that the differences $K_i(m+\Delta m,n+\Delta n)-K_i(m,n)$ can be expanded in Taylor series. Eliminating the p_i from Eqs. (7) and expanding to linear order thus leads to the differential transformation equations[9]

$$\partial K_i/\partial n=\textstyle\sum_j D_{ij}(K_1,K_2)\,\partial K_j/\partial m,\quad i=1,2. \qquad (11)$$

The coefficients $D_{ij}(K_1,K_2)=(-1)^i(\partial K_i/\partial p_i)(\partial p_i/\partial K_j)$ are highly nonlinear in the K_i. The boundary conditions are $K_2(0,n)=0$ and $K_i(\infty,n)=K_{iB}$. Equations (11) together with their boundary conditions give a full and exact description of all sufficiently smooth solutions of the recursion of Eqs. (7).

One can indeed find explicitly a family of similarity solutions $[\tilde{K}_1(m/n),\tilde{K}_2(m/n)]$ to Eq. (11). For one of these (the "critical" similarity solution) $\tilde{K}_1$ and $\tilde{K}_2$ satisfy the criticality condition at all m/n. Setting $m=\tfrac{1}{2}$ in this solution[10] we find *analytically* that $\exp[-4K_1(\tfrac{1}{2},n)]\simeq 1/2n$, which via Eqs. (9) and (10) leads to the result $\eta=1$ of Eq. (1).

One might be tempted to treat the inhomogeneous system of Eq. (2) by linearizing around the critical similarity solution. It turns out that this is correct for $y>1$, but that for $0<y\le 1$ one has to study the nonlinear equations (11). We solve these exactly. After a transformation from the unknowns $K_1(m,n)$ and $K_2(m,n)$ to new ones $u(m,$

$n)$ and $v(m, n)$, and after substantial algebra, the equations take a convenient form. The transformation reads

$$\sinh 2K_1 = [(v^2 - u^2)/4uv]\, Q(u, v), \tag{12}$$

$$\sinh 2K_2 = \frac{1}{v} Q(u, v), \tag{13}$$

$$Q(u,v) = \left[1 - \left(\frac{v-u}{2}\right)^2\right]^{1/2} - \left[1 - \left(\frac{v+u}{2}\right)^2\right]^{1/2}. \tag{14}$$

Equations (12)–(14) map the supercritical domain onto the triangle $0 < u < v$, $u + v < 2$. Criticality corresponds to $u + v = 2$. Finally we perform the so-called hodograph transformation,[11] i.e., we consider m and n as functions of u and v. The equations for $m(u,v)$ and $n(u,v)$ are linear and read

$$v\partial m/\partial u = u\partial n/\partial v, \quad v\partial m/\partial v = u\partial n/\partial u. \tag{15}$$

The boundary conditions stated above become $u(0,n) = 0$, $u(\infty,n) = u_B$, and $v(\infty,n) = v_B$ (the index indicating bulk values). Equations (15) are solved by separation of variables. The solutions are (modified) Bessel functions I_n and K_n with $n = 0, 1$. The boundary conditions lead us to consider superpositions with weight $w(\rho)$ of the type

$$m(u,v) = u\int_0^\infty d\rho\, w(\rho) \exp(-\rho u_B) I_1(\rho u)[K_1(\rho v_B) I_0(\rho v) + I_1(\rho v_B) K_0(\rho v)],$$
$$n(u,v) = v\int_0^\infty d\rho\, w(\rho) \exp(-\rho u_B) I_0(\rho u)[K_1(\rho v_B) I_1(\rho v) - I_1(\rho v_B) K_1(\rho v)]. \tag{16}$$

These are exact solutions of the flow problem (11). The results (4)–(6) follow from analysis of Eqs. (16) for $m \to \infty$ at fixed n (regime I) and for $n \to \infty$ at fixed m (regime II). The weight function $w(\rho)$ is prescribed by matching it to the initially given system (2) in regime I. It determines in turn the asymptotic behavior in regime II, from which we have to extract our results.

To make contact with Eq. (2) we let $w(\rho) \simeq C\rho^{\sigma+1/2}$ for $\rho \to \infty$. The analysis in regime I shows that then $\delta K_i \simeq -A_i/m^y$ provided we choose $\sigma = 1/2y$ and $C = 4\pi^{-1/2}A^{1/y}/\Gamma(1/2y)\sinh 4K_{2B}$. In regime II, for small m, u must be small because of the $m = 0$ boundary condition. For a system close to criticality we can therefore put $v = 2 - \delta v$, with δv small. Asymptotic expansion in regime II then gives $u(m,n)$ and $\delta v(m,n)$ for $n \to \infty$. From Eqs. (12) we find, by substituting[10] $2 - \delta v(\frac{1}{2}, n)$ and $u(\frac{1}{2}, n)$, the desired quantity $\exp[-4K_1(\frac{1}{2}, n)]$ for $n \to \infty$. Via Eqs. (9) and (10) this leads to our results (4)–(6).

We remark that for $n \to \infty$ the solution (16) can be shown to approach the similarity solution mentioned above. The fact that only the asymptotic behavior of $w(\rho)$ enters our considerations implies a universality property for inhomogeneous lattices. Finally, the theory presented here can also be used to calculate, in principle, other boundary correlations and the boundary magnetization. Deviations from homogeneity towards the low-temperature side ($A_i < 0$) have not been discussed here: These require different $m = 0$ boundary conditions for the differential equations. Details will be published elsewhere.

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[2]This result, formulated in Ref. 1 for a square lattice, also holds for a triangular lattice. The two are interconverted by a lattice restructuring transformation [R. J. Baxter and I. G. Enting, J. Phys. A 11, 2463 (1978)].

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[7]The star-triangle relation can be cast in other forms (see, e.g., Ref. 6), some of which are calculationally more convenient.

[8]We thank Professor P. Hemmer for a discussion on the derivation of Eq. (9).

[9]Equations (11) are related to, but different from, the exact renormalization group equations for the Ising model. See H. J. Hilhorst, M. Schick, and J. M. J. van Leeuwen, Phys. Rev. 19, 2749 (1979).

[10]The procedure of recovering the lattice variable $K_1(\frac{1}{2}, n)$ from the continuum description may seem to depend on the labeling convention for the bonds. To justify it we put $K_i(m,n) = K_i^0(m,n) + K_i^1(m,n)$ where K_i^0 satisfies Eq. (11). We substitute this in Eq. (7) and linearize in K_i^1. In general an inhomogeneous equation for the K_i^1 is expected, but with our particular labeling it becomes homogeneous. Hence the K_i^0 do not drive any corrections K_i^1, to linear order. We *assume* here that the higher-order corrections are negligible.

[11]See, e.g., W. F. Ames, *Nonlinear Partial Differential Equations in Engineering* (Academic, London, 1965).