# Approaching the Quantum "Limit" for Force Detection

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(Received 7 August 1981)

A system is proposed for detecting the effect of a weak classical force on a macroscopic mechanical oscillator. The novel feature is a type of ac bridge coupled to a SQUID amplifier. A detailed sensitivity calculation for one type of bridge excitation is reported which shows what is required to reach the standard quantum "limit." Another type of excitation should make it a "back-action evasion" device capable of exceeding the standard quantum "limit".

PACS numbers: 06.30.Gr, 03.65.-w, 07.50.+f

At what level of sensitivity will one reach a fundamental limit for the measurement of a weak impulsive classical force? This question has arisen from the effort to detect gravity waves from astronomical sources.[1] If the force is sensed by its effect on a mechanical oscillator with angular frequency $\omega_1$ in an initially unexcited state, the Heisenberg uncertainty principle *seems* to imply a lower limit to the detectable force, corresponding to an impulse which changes the oscillator's energy by about $\hbar\omega_1$. Caves *et al.*,[2] Hollenhorst,[3] and Unruh[4] have reviewed the arguments that lead to such a "standard quantum limit," have concluded that this limit is not fundamental, and have described newly invented measurement strategies which, in principle, will exceed this limit. These measurement strategies are called "back-action evasion" or "quantum nondemolition."

We propose below an oscillator-transducer-amplifier system which should be able to compare directly three distinct conventional measurement strategies and a back-action evasion strategy. Therefore it is capable of addressing questions in quantum measurement theory experimentally. It also looks attractive as part of a gravity-wave detector. In passing, we note that the electrical circuit is potentially more sensitive than conventional electronics for measuring any quantity which can be made to vary a capacitance.

After defining the system, we report on a detailed classical sensitivity calculation for one conventional measurement strategy or mode of operation. It shows that it is feasible to reach the standard quantum limit,[5] which is the minimal precondition for testing quantum measurement theory. The other modes and the role of back-action evasion will be discussed in a future publication.

The system is based on the "continuous single transducer" measurement scheme of Caves *et al.*[2] (see also Thorne *et al.*[6]). The mechanical oscillator can be realized by an accelerometer of natural frequency $\omega_1$ as shown in Fig. 1(a). Attached are three capacitor plates ($V_1$, $V_3$, and $V_2$), each of area $A$, separated by a mean gap $D$. A force $F(t)$, applied to the outer plates (mass $m$), causes a relative displacement $x(t)$ of the inner and the outer plates.

This motion is sensed by the electrical circuit shown in Fig. 1(b). It is clearly a type of ac bridge. A low-input-impedance amplifier (a SQUID) detects the bridge imbalance.[7] The unusual feature of the bridge is the electrical resonance, chosen to be at some frequency $\omega_2$, between the detector arm ($L_R + L_I$) and the capaci-

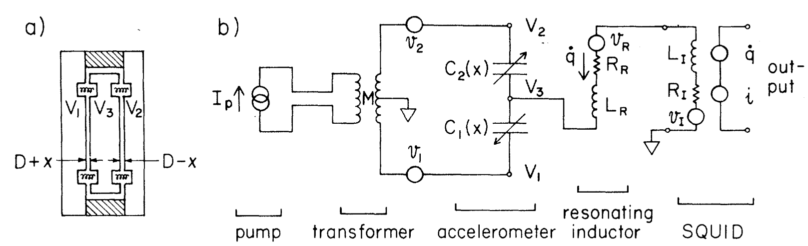


FIG. 1. (a) A schematic diagram of an accelerometer. An impulse applied to the outer plates causes the relative coordinate $x$ to oscillate. (b) The electrical schematic of the bridge circuit used to sense the motion of $x$. Noise generators are labeled with italic letters.

 

tors. Johnson noise caused by lossy circuit elements is represented by the voltage generators $v_1$, $v_2$, and $v_R$. Following Giffard and co-workers,[8] the noise in the SQUID is represented by (1) a series input noise-voltage generator $v_I(t)$, and (2) an equivalent additive noise generator $\dot{i}(t)$, referred to the input current. In this paper we shall assume that the bridge exitation $I_p$ is noiseless.

To a good approximation the capacitances are given by

$$C_{1,2}(x)^{-1} = C_0^{-1}[1 \pm x(t)/D] = (D/\epsilon_0 A)[1 \pm x(t)/D].$$

Then, using standard circuit theory and the following definitions (see also Fig. 1): $V_p(t) \equiv \frac{1}{2}[V_2(t) - V_1(t)] \cong M\dot{I}_p(t)$, $v(t) \equiv \frac{1}{2}[v_2(t) + v_1(t)] + v_R(t) + v_I(t)$, $L \equiv L_R + L_I$, $R \equiv R_R + R_I$, $\omega_2 \equiv (2C_0L)^{-1/2}$, $\tau_2 \equiv Q_2/\omega_2 \equiv L/R$, we arrive at the classical equations of motion for the mechanical variable $x(t)$ and the current $\dot{q}(t)$[9]:

$$\ddot{q}(t) + \tau_2^{-1}\dot{q}(t) + \omega_2^2 q(t) = -(LD)^{-1} V_p(t)\, x(t) + L^{-1} v(t), \tag{1}$$

$$\ddot{x}(t) + (\omega_1/Q_1)\,\dot{x}(t) + \omega_1^2 x(t) = -(\mu D)^{-1} V_p(t)\, q(t) + m^{-1}F(t) + \mu^{-1} f(t), \tag{2}$$

where we have dropped terms of higher order than first in $x(t)/D$; $\mu$ is the reduced mass and $f(t)$ is the Langevin force responsible for Brownian motion.

The mechanical and electrical oscillators are coupled by a controllable time-dependent voltage $V_p(t)$.[10] We can enumerate at least four different interesting modes of operation for the device: (a) resonant bridge, $V_p(t) = V_0 \cos\omega_2 t$; (b) mechanical-to-electrical parametric upconverter, $V_p(t) = V_0 \cos(\omega_2 - \omega_1)t$; (c) mechanical parametric amplifier, $V_p(t) = V_0\cos(\omega_2 + \omega_1)t$; and (d) back-action–evasion device, $V_p(t) = V_0 \cos\omega_1 t \times \cos\omega_2 t$. As indicated by their labels, modes (b) and (c) are electromechanical parametric processes that are directly analogous to well known (purely) electronic parametric processes.[11] Mode (d) is a way to realize the "continuous single transducer" measurement scheme of Caves *et al.*[2] Throughout the remainder of this paper we will restrict our attention to mode (a), which is an intermediate case not ordinarily discussed.

The sensitivity of the system is calculated in four steps:

First, we find an approximate solution for the Fourier transform of the mechanical variable, $\tilde{x}(\omega)$, by transforming Eq. (2) and substituting into the transform of Eq. (1). Assuming $\omega_2 \gg \omega_1$, we may drop the terms $\tilde{x}(\omega \pm \omega_2)$ and other terms cancel to first order. The result, peculiar to mode (a), is that the response to an external force is unaffected by the coupling, the only change being the addition of a back-action noise force proportional to $\tilde{v}$.

Second, with $\tilde{x}(\omega)$ determined, we solve the transform of Eq. (1) for the observed output, $j\omega\tilde{q}(\omega) + \tilde{i}(\omega)$. It is convenient to express this result as a quantity $\tilde{X}(\omega)$, which we call the apparent displacement. It is extracted by synchronously demodulating the output and multiplying the low-frequency components by $[(V_0/2LD)j\omega_2 H_2(\omega)]^{-1}$, where $j \equiv \sqrt{-1}$ and $H_2(\omega) \equiv [\omega_2(-2\omega + j/\tau_2)]^{-1}$.

Third, we calculate the "signal" and the "noise" content of the apparent displacement. The signal is $\tilde{X}_F(\omega) = G_1(\omega)\tilde{F}(\omega)/m$, where $G_1(\omega) \equiv [-\omega^2 + j(\omega_1/Q_1)\omega + \omega_1^2]^{-1}$ is the usual harmonic oscillator response function. The noise has a power spectral density

$$S_X(\omega) = |G_1(\omega)|^2 \left\{\frac{1}{\mu^2} S_f(\omega) + \frac{2}{\mu^2}\left(\frac{V_0}{2LD}\right)^2 |H_2(\omega)|^2 S_v(\omega_2)\right\} + \frac{1}{2}\left(\frac{2LD}{V_0\omega_2}\right)^2 |H_2(\omega)|^{-2} S_i(\omega_2) + 2\left(\frac{D}{V_0}\right)^2 S_v(\omega_2). \tag{3}$$

Fourth, we filter the apparent displacement for optimal detection of an impulse applied at a *known* time. By standard methods we are able to calculate the mean square noise-equivalent impulse $\langle p_n^2\rangle$ of the optimally filtered signal. We express this final result in units of the energy quantum by defining the impulse noise number $n_I$,[12]

$$n_I \hbar\omega_1 \equiv \frac{\mu}{2m^2}\langle p_n^2\rangle = \frac{\mu}{2}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|G_1(\omega)|^2}{S_X(\omega)} d\omega\right\}^{-1}. \tag{4}$$

This expression was evaluated by finding the poles of the integrand with a computerized iterative Newton-Raphson technique and then summing the residues.[13]

The results of this calculation of $n_I$ can be shown to be a function of only four dimensionless ratios of the physical parameters: the total electrical noise number, $n_E \equiv [S_v(\omega_2)S_i(\omega_2)]^{1/2}/\hbar\omega_2$; $\alpha \equiv (k_BT/\hbar\omega_1 Q_1)/n_E$; $\beta \equiv (V_0/D)^2\,(\omega_2/\omega_1)\,(C_0/\mu\omega_1^2)$; and $\gamma \equiv (\omega_2/\omega_1)\,(\omega_2 C_0)\,[S_v(\omega_2)/S_i(\omega_2)]^{1/2}$. We can show that if $Q_2 \gg (\alpha/\gamma)Q_1$, the circuit contribution to

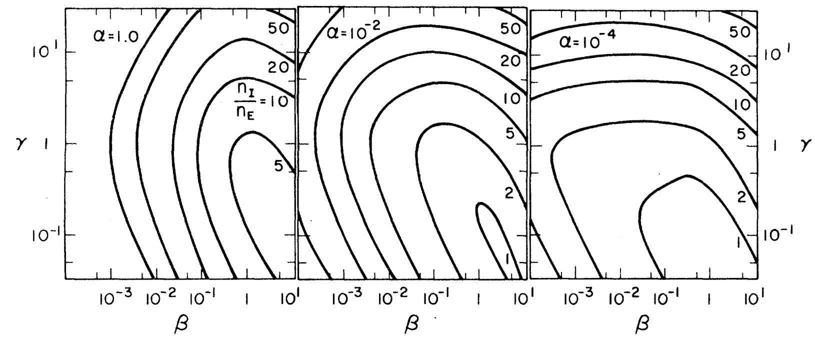


FIG. 2. The result of our calculation of $n_I$, the overall system noise number for the detection of an impulse. It depends on four dimensionless coefficients: The total electrical noise number, $n_E$; a quantity proportional to the ratio of mechanical to electrical noise power, $\alpha$; an electromechanical coupling coefficient, $\beta$; and the ratio of total noise impedance to source impedance, $\gamma$.

$n_E$ becomes small and $n_E \to n_A$, the analogous amplifier noise number.

In Fig. 2 we have plotted the results as contours of constant noise number ratio in the $\beta$-$\gamma$ plane, for three values of $\alpha$. An important result is that the minimum attainable noise number ratio is 1. For $Q_2$ large enough and a quantum limited amplifier,[14] this corresponds to the standard quantum limit. The results of our detailed model then agree with the ultimate sensitivity limits derived by Giffard[15] for a generalized *linear* motion detector.

Table I lists two examples of the parameters that would achieve the standard quantum limit. Case 1 represents a device suitable for a small-scale test; Case 2 represents a large mass system that might be the accelerometer on the end of a larger gravity-wave antenna. Most of the critical physical parameters have either been achieved ($Q_1$, $V_0/D$) or approached ($Q_2$) individually in roughly comparable circumstances.

Quantum-limited SQUID's are not yet available but progress is rapid and the prospects seem good.[16] Using the approximate theory of Tesche and Clarke[17] and the measured output noise of the well coupled SQUID of Cromar and Carelli[18] we infer a noise number $n_A \sim 28$ and a noise resistance $R_N \equiv (S_{vI}/S_i)^{1/2} \sim 1.4\times10^{-2}\ \Omega$ at 23 kHz. This theory also predicts how the noise changes with frequency: $S_i(\omega)$ is frequency independent and $S_{vI}(\omega)$ is proportional to $\omega^2$. Thus noise resistance $R_N$ is proportional to $\omega$, and noise number $n_A$ is frequency independent. The noise resistance assumed in Case 1 of Table I is nearly that of Cromar and Carelli, corrected for the different frequency.

Another conclusion of this analysis is that the best SQUID design for this transducer would be different from what is now thought of as a useful SQUID, because a much smaller input inductance

TABLE I. Two possible choices of physical parameters (and derived quantities) for the resonant bridge mode, which would achieve the standard quantum "limit" for measurement of a classical impulsive force. The symbols are defined in the text.

| | Case 1 | Case 2 |
|---|---|---|
| | Mechanical oscillator | |
| $\mu$ | 0.05 kg | 50 kg |
| $\omega_1/2\pi$ | 1 kHz | 1 kHz |
| $Q_1$ | $2\times10^9$ | $2\times10^9$ |
| $T$ | 0.14 K | 0.14 K |
| | Transducer | |
| $A$ | $7\times10^{-4}$ m$^2$ | $7\times10^{-2}$ m$^2$ |
| $D$ | $1\times10^{-5}$ m | $1\times10^{-5}$ m |
| $\omega_2/2\pi$ | 0.28 MHz | 0.60 MHz |
| $V_0$ | 15 V | 30 V |
| $C_0$ | $6.2\times10^{-10}$ F | $6.2\times10^{-8}$ F |
| $L_R$ | $2.6\times10^{-4}$ H | $5.7\times10^{-7}$ H |
| $Q_2$ | $>1\times10^7$ | $>1\times10^7$ |
| $V_0/D$ | $1.5\times10^6$ V/m | $3.0\times10^6$ V/m |
| | Amplifier | |
| $S_{vI}(\omega_2)$ | $1.5\times10^{-28}$ V$^2$/Hz | $7.0\times10^{-31}$ V$^2$/Hz |
| $S_i(\omega_2)$ | $4.6\times10^{-28}$ A$^2$/Hz | $4.5\times10^{-25}$ A$^2$/Hz |
| $L_I$ | $1.6\times10^{-6}$ H | $1.6\times10^{-9}$ H |
| $n_A$ | $(\ln 2)^{-1}$ | $(\ln 2)^{-1}$ |
| $R_N(\omega_2)$ | 0.57 Ω | $1.2\times10^{-3}$ Ω |
| | Dimensionless parameters | |
| $\alpha$ | 0.001 | 0.001 |
| $\beta$ | 0.2 | 0.2 |
| $\gamma$ | 0.2 | 0.2 |
| $n_I/n_A$ | 1 | 1 |

$L_I$ is desirable. The approximate theory[17] predicts that (at a fixed frequency) $n_A$ is independent of $L_I$ and $R_N$ is proportional to $L_I$. Case 2 of Table I illustrates how we might use this to our advantage. By an increase in $C_0$, a decrease in $L$, and reduction of $L_I$ to get the proper $R_N$, it is possible to maintain good impedance matching and good coupling to a much larger mass with only a modest increase in field.

We especially thank Carl Caves for many valuable conversations and Dave Douglass for his help and encouragement. We also received helpful remarks from M. W. Cromar, W. Hamilton, H. Padamsee, W. G. Unruh, and the Cryoelectronics Group at the National Bureau of Standards, and help on the computations from J. R. Marsden. This work was initiated at the Aspen Center for Physics, and supported by the National Science Foundation under Grant No. PHY-8008896.

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[2]C. M. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmerman, Rev. Mod. Phys. 52, 341 (1980).

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[5]Paik has discussed approximately the conditions required for a type of system that cannot be operated in back action evasion mode. H. J. Paik, Nuovo Cimento 55B, 15 (1980).

[6]K. S. Thorne, C. M. Caves, V. D. Sandberg, M. Zimmermann, and R. W. P. Drever, in *Sources of Gravitational Radiation*, edited by Larry Smarr (Cambridge Univ. Press, London, 1979), p. 49.

[7]Common usage for SQUIDs is at audio frequencies, but higher frequency usage is not unknown, see, e.g., J. E. Zimmerman and D. B. Sullivan, Appl. Phys. Lett. 31, 360 (1977).

[8]M. B. Simmonds, W. A. Fertig, and R. P. Giffard, IEEE Trans. Magn. 15, 478 (1979). See also J. Clarke, C. D. Tesche, and R. P. Giffard, J. Low Temp. Phys. 37, 405 (1979).

[9]The effect of the transformer on the readout circuit may be ignored if the transformer secondary coils are well matched and coupled very strongly to one another.

[10]There are many similarities between this circuit and various microwave devices under development for use as transducers on gravity wave detectors. See, e.g., W. C. Oelfke, W. O. Hamilton, and D. Darling, IEEE Trans. Magn. 17, 853 (1981); D. G. Blair, in *Gravitational Radiation, Collapsed Objects and Exact Solutions*, Lecture Notes in Physics Vol. 124 (Springer-Verlag, Berlin, 1980), p. 314. A detailed comparison of all these devices will require a more elaborate calculation.

[11]William H. Louisell, *Coupled Mode and Parametric Electronics* (Wiley, New York, 1960).

[12]If the impulse arrives at an unknown time, and if the optimum filter time constant is > a few periods, we can show that the traditional measure of gravity wave sensitivity, called the burst temperature $T^*$, is given by $T^* = 2n_I\hbar\omega/k_B$.

[13]This method was brought to our attention by a preprint of P. F. Michelson and R. C. Taber.

[14]H. Heffner, Proc. Inst. Radio Electron. Eng. 50, 1604 (1962).

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[16]For a recent review, see M. B. Ketchen, IEEE Trans. Magn. 17, 387 (1981).

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