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Simple Derivation of the Baxter-Model Free Energy

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Starting with some observations due to A. B. Zamolodchikov, a short cut to the determination of the Baxter (eight-vertex) -model free energy is developed. The method involves factorizable S matrices in $1+1$ dimensions, which are briefly reviewed. If the method generalizes, we may associate with each known S matrix a Baxter-like model with a *known* free energy.

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In this paper a novel connection between two seemingly unrelated problems is explained. The first problem is the determination of the free energy of Baxter's eight-vertex model[1] briefly discussed below. The second problem is the determination of the S matrix in $1+1$ dimensions which has Z_4 symmetry (i.e., obeys charge conservation modulo 4), is Lorentz invariant, and is elastic and *factorizable* (i.e., the multiparticle S matrices are products of two-body S matrices, one for each two-body encounter). It was solved by Zamolodchikov[2] using certain general principles such as unitarity, crossing, etc., as well as constraints imposed by the factorizability requirement.[3] Zamolodchikov noticed, on solving for the S matrix, that one of his S-matrix elements was very simply related to the free energy determined by Baxter for the eight-vertex model. *Here this coincidence is explained,* i.e., it is shown that the knowledge of the two-body S matrix is indeed equivalent to the knowledge of the free energy and that the two are related in a manner that Zamolodchikov noticed. The present connection between an *on-shell* S matrix and a statistical problem on a lattice and *not necessarily critical* is very different from the usual one between Euclidean Green's functions of quantum field theory and the correlation function of a critical statistical system.

In the eight-vertex problem Baxter considers an $N\times N$ lattice on the bonds of which are drawn arrows subject to the requirement that only the four vertices in Fig. 1, with Boltzmann weights a, b, c, and d, and four more with the same weights but all arrows reversed, are allowed. He parametrizes the ratios of weights as follows:

$$a:b:c:d=\frac{\mathrm{sn}2\eta(1+i\,\theta/\pi)}{\mathrm{sn}2\eta}:\frac{-\mathrm{sn}2\eta i\theta/\pi}{\mathrm{sn}2\eta}:1:k\,\mathrm{sn}\,\frac{2\eta i\theta}{\pi}\,\mathrm{sn}2\eta\left(1+\frac{i\theta}{\pi}\right), \tag{1}$$

where sn are Jacobi elliptic functions of modulus k with periods (for $0<k<1$) $4K$ and $2iK'$, where K and K' are complete elliptic integrals of modulus k and k'.[4] [Actually Baxter used a parameter $\nu=\eta(1+2i\theta/\pi)$.] He got the partition function per site in the limit $N\to\infty$ in the principal region (PR), which corresponds to $c>a+b+d$, all positive, or $0<\mathrm{Im}\,\eta<\frac{1}{2}K'$, $0<\mathrm{Im}\,\theta<\pi$, with η and θ pure imaginary. Using the Fan and Wu identities[5] one could get the answer everywhere else. We come to this answer in a moment.

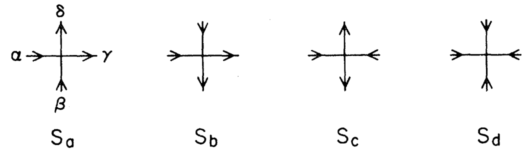

FIG. 1. The four independent amplitudes in two-body scattering.

The Z_4-symmetric S matrix in $1+1$ dimensions[3] describes collisions between particles (p) and antiparticles ($\bar{p}$) whose charge is conserved modulo 4, so that $pp \to \bar{p}\bar{p}$ is allowed. In a two-body collision there are eight types of charge states, and eight amplitudes. Four of these are depicted in Fig. 1, while four others, equal to these and related by reversal of all arrows, are not. For example, S_c describes a reaction $p(\theta/2)+\bar{p}(-\theta/2) \to \bar{p}(\theta/2)+p(-\theta/2)$, where $\pm\theta/2$ are c.m. rapidities. Note the natural correspondence between the processes in Fig. 1 and the eight vertices of Baxter. (S_a corresponds to a, etc.) Factorizability demands that[2]

$$S_a : S_b : S_c : S_d = a:b:c:d. \tag{2}$$

The S_i are required to be meromorphic; to obey the crossing relations

$$S_c(\theta)=S_c(i\pi-\theta), \quad S_d(\theta)=S_d(i\pi-\theta), \\ S_a(i\pi-\theta)=S_b(\theta); \tag{3}$$

to be real analytic (i.e., be real on the $\mathrm{Im}\,\theta$ axis),

$$S_i(-\theta^*)=S_i^*(\theta); \tag{4}$$

and to satisfy unitarity,

$$S(\theta)S^T(-\theta)=I. \tag{5a}$$

Equation (5a) constrains $S_c(\theta)$ [on eliminating others using Eq. (1)] as follows:

$$S_c(\theta)S_c(-\theta)=\frac{\mathrm{sn}^2 2\eta}{\mathrm{sn}^2 2\eta - \mathrm{sn}^2(2\eta i\theta/\pi)}. \tag{5b}$$

This equation has a unique "minimal" solution[2] if we require it to be free of poles and zeros in the "physical strip" $0<\mathrm{Im}\,\theta<\pi$. It obtains for $0<\mathrm{Im}\,\eta<K'(k)/2$, $0<k<1$, and is given by

$$S_c(\theta)=\exp\left\{4\sum_{n=1}^{\infty}\frac{\sinh^2\left(\dfrac{2\pi n(\pi-\gamma)}{\gamma'}\right)\sin\left(\dfrac{2\pi n\theta}{\gamma'}\right)\sin\left(\dfrac{2\pi n(i\pi-\theta)}{\gamma'}\right)}{n\sinh\left(\dfrac{4\pi n\gamma}{\gamma'}\right)\cosh(2\pi^2 n/\gamma')}\right\}, \tag{6}$$

where $\gamma = i\pi K'/2\eta i$ and $\gamma' = 2\pi K i/\eta$. Zamolodchikov observed the remarkable coincidence that the Baxter-model free energy per site f, *in the* PR, is related to S by

$$z=\exp[-\beta f(a,b,c,d)]=\frac{c(\theta,\eta,k)}{S_c(\theta,\eta,k)} \\ =\frac{a}{S_a}=\frac{b}{S_b}=\frac{d}{S_d}. \tag{7}$$

We now proceed to explain this coincidence by proving the following equivalent result.

Theorem.—The partition function per site, $z(\theta,\eta,k)$, *is unity in the PR if* $S_a,\ldots,S_d$ *are used as vertex weights.* (Note that $S_a,\ldots,S_d$ are special cases of $a,\ldots,d$, normalized by unitarity and constrained by crossing.)

Proof.—Consider the process where a projectile with rapidity θ and internal state i_1 ($=p$ or $\bar{p}$) collides with N targets at rest and in an internal state given by a collective label α_1. Let i_1' and α_2 be the final states. Fig. 2(a) is a space-time picture of this process. Let us denote by $S_{i_1,\alpha_1}{}^{i_1',\alpha_2}(\theta)$ the $(N+1)$-body S-matrix elements. (Hereafter S shall stand for this S matrix. The two-body S matrix will be referred to by name.) S is just the monodromy matrix of Takhtadzhan and Faddeev[6] and is related to Baxter's transfer

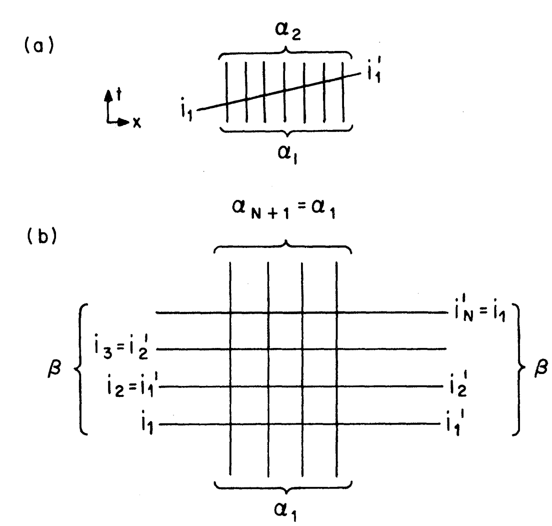

FIG. 2. (a) A space-time picture of a collision between a particle of rapidity θ in (internal) state i_1 with N particles at rest in state α_1. The final states are i_1' and α_2. (b) The skew-periodic lattice.

matrix T by

$$T_{\alpha_1,\alpha_2}=\sum_{i_1,i_1'}(S_{i_1\alpha_1}{}^{i_1'\alpha_2})\delta_{i_1 i_1'}. \tag{8}$$

The Baxter partition function with toroidal boundary condition on an $N\times N$ lattice is, in the PR *(to which we restrict ourselves hereafter)*,

$$Z_B=\operatorname{Tr}T^N$$
$$=\Lambda_B{}^N(1+\text{terms vanishing as } N\to\infty), \tag{9}$$

where $\Lambda_B{}^N$ is the largest eigenvalue of T in the PR.

Consider now an $N\times N$ lattice periodic in the vertical direction, but skewed in the horizontal direction—the right-most state in row n (i_n') equals the left most in row $n+1$ (i_{n+1}) [see Fig. 2(b)]. Let $\beta=(i_1,i_2,\ldots,i_N)$ and $\beta'=(i_1',\ldots,i_N')$. The reason behind introducing this skewed partition function Z_s is that

$$Z_s=\operatorname{Tr}S^N$$
$$=2\Lambda_s{}^N(1+\text{terms vanishing as } N\to\infty), \tag{10}$$

where Λ_s is the largest eigenvalue of S. (Even though S is not diagonalizable in the PR, it is made up of two identical, positive blocks, each of the form $\Lambda_s Z^{-1}PZ$, where P is a stochastic matrix whose trace $\to 1$ as $N\to\infty$.[7]) Viewing Fig. 2(b) from the side, we get

$$Z_s=\sum_\beta[T^N]_{\beta\beta'}$$
$$\simeq\Lambda_B{}^N\sum_\beta\langle\beta|\Lambda_B\rangle\langle\Lambda_B|\beta'\rangle=\Lambda_B{}^N C_N, \tag{11}$$

where $|\Lambda_B\rangle$ is the eigenket that goes with Λ_B. (If $|\Lambda_B|$ is asymptotically degenerate C_N is suitably modified.) Equations (10) and (11) tell us that

$$\Lambda_s(\theta)=\Lambda_B(\tfrac{1}{2}C_N)^{1/N} \tag{12}$$

(in the PR). Note that since $|\Lambda_B\rangle$ is θ independent, so is C_N and that $\Lambda_B(\theta)=\langle\Lambda_B|T(\theta)|\Lambda_B\rangle$ is meromorphic as the weights are.

Our motivation for deriving Eq. (12) is the following: We shall shortly show that *the analytic continuation of $\Lambda_s(\theta)$ from $0<\mathrm{Im}\theta<\pi$ to $-\pi<\mathrm{Im}\theta<0$, along the $\mathrm{Im}\,\theta$ axis, obeys*

$$\Lambda_s(-\theta)\Lambda_s(\theta)=1. \tag{13}$$

(Once established on this line segment, the relation holds in the entire θ plane.) Equation (12) implies then that

$$\tilde\Lambda_B(-\theta)\,\tilde\Lambda_B(\theta)=(\tfrac{1}{2}C_N)^{-2/N}, \tag{14}$$

where $\tilde\Lambda_B$ is defined for all θ as Λ_B minus those terms that vanish in the $N=\infty$ limit in the PR. We are then led to the following:

$$z_B(\theta)z_B(-\theta)=\lim_{N\to\infty}\tilde\Lambda_B{}^{1/N}(\theta)\tilde\Lambda_B{}^{1/N}(-\theta)$$
$$=(\tfrac{1}{2}C_N)^{-2/N^2}=1. \tag{15}$$

[If $C_N{}^{1/N^2}$ did not tend to 1 as $N\to\infty$, Eq. (12) would imply that the free energy per site with skewed boundary conditions does not equal that with toroidal boundary conditions. We exclude this unphysical possibility.] Note that $z_B(\theta)$ equals z, the true partition function per site, in the PR, but not necessarily elsewhere in the θ plane.

Let us now turn to the proof of Eq. (13). Since $S^T(-\theta)=S^{-1}(\theta)$, $\Lambda_s{}^{-1}(\theta)$ is an eigenvalue of $S^T(-\theta)$. It is also nondegenerate and the smallest in modulus. Since S and S^T have the same spectrum, *Eq. (13) boils down to the claim that the analytic continuation of the largest eigenvalue of $S(\theta)$ from $\mathrm{Im}\theta>0$ to $\mathrm{Im}\theta<0$ gives the smallest eigenvalue of S there.* To explore the crossover, we expand $S(\theta)$ near $\theta=0$ and work to order θ:

$$S(\theta)=S_0+\theta S_1. \tag{16}$$

To first order, we have in obvious notation

$$\Lambda_i(\theta)=\Lambda_i(0)+\theta\langle\Lambda_i^0|S_1|\Lambda_i^0\rangle.$$

At $\theta=0$, unitarity of S_0 says $\Lambda_i(0)$ lie on the unit circle. For θ real, unitarity of $S(\theta)$ requires the first order changes to be perpendicular to $\Lambda_i(0)$ in the complex plane so that

$$\Lambda_i(\theta)=\Lambda_i(0)[1-i\theta\Delta_i]$$
$$\simeq\Lambda_i(0)\exp(-i\theta\Delta_i). \tag{17}$$

In the PR, where $\theta=i|\theta|$, we must have $\Delta_s>\Delta_i$ in order that Λ_s dominate. By the same token, as we move down the $\mathrm{Im}\theta$ axis to $\theta=-i|\theta|$, we see that Λ_s smoothly evolves into the *smallest* eigenvalue. (I have verified that degeneracies do not invalidate this argument.)

Returning to z_B, the relation $Z(S_a,S_b,S_c,S_d)=Z(S_b,S_a,S_c,S_d)$ and the crossing equations (3) imply that

$$z_B(\theta)=z_B(i\pi-\theta) \tag{18}$$

(Actually, this is valid for z, but $z=z_B$ in the PR and *$\theta\to i\pi-\theta$ maps the PR into itself.*) Finally, the fact that under $\theta\to\theta+\gamma'/2$ [where $\gamma'=2\pi iK(k)/\eta$], some weights at the most change sign and T is invariant tells us that

$$z_B(\theta)=z_B(\theta+\gamma'/2). \tag{19}$$

Equations (15) and (18) imply that $z_B(-\theta)=1/$

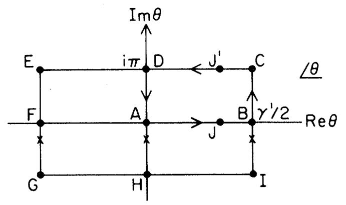

FIG. 3. A portion of the θ plane. Poles of the two-body S matrix occur on AH, BI, FG, etc. and are shown by crosses. $ADCBIH$ is the fundamental period of z_B. On AD, $z_B = z$, the true partition function per site.

$z_B(\theta) = 1/z_B(i\pi - \theta)$ or that

$$z_B(\theta) = z_B(2\pi i + \theta). \qquad (20)$$

Thus we need to study z_B just in one period, say the rectangle $ADCBIH$ in Fig. 3. Since $\tilde{\Lambda}_B(\theta)$ is meromorphic, $z_B(\theta)$ can be singular only at a pole or zero of $\tilde{\Lambda}_B(\theta)$. Poles of $\tilde{\Lambda}_B$ are traceable to poles of the two-body S matrix, which are known to lie only on AH and BI,[2] and are given by crosses in Fig. 3. But since $\Lambda_B(\theta)$ is never zero on AD it cannot have a pole on AH. As for a zero of $\tilde{\Lambda}_B(\theta)$ in $ADCBIH$, it would imply a pole in $AHGFED$ which is impossible. (More precisely z_B cannot vanish on $ADCBIH$ because it cannot blow up in $AHGFED$.) Thus z_B is a constant by Liouville's theorem and equal to 1 because $z_B(0)^2 = 1$.

There exist two other short cuts, all based on determining z_B from some functional relations. The one due to Baxter[8] is essentially the same (though his face transfer matrices are not normalized by unitarity). His Eqs. (3.28) and (3.29) coincide with my Eqs. (5b) and (3) on setting $u = -\theta\eta/K$ and $\lambda = \pi\eta/iK$. But he has to assume his crucial Eq. (3.31), while I do not. In Stroganov's approach,[9] the inversion formula [his Eq. (13)] is shown to be valid for θ near zero (where T is close to a shift operator) but assumed to hold for all θ. (See Ref. 10 for other related works.) The above theorem puts these earlier schemes on a firm footing in the Z_4–eight-vertex case. To do the same for other models, such as the nineteen-vertex model of Zamoldchikov and Fateev,[11] one must verify that the ingredients for a similar theorem are present.

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[1]R. J. Baxter, Ann. Phys. 70, 193 (1972), and J. Stat. Phys. 8, 25 (1973).

[2]A. B. Zamolodchikov, Commun. Math. Phys. 69, 165 (1979).

[3]For a comprehensive review of factorizable S matrices in 1+1 dimensions, see A. B. Zamolodchikov and Al. B. Zamolodchikov, Ann. Phys. 120, 253 (1979). For a simple review see R. Shankar, Yale University Report No. C00-3075-199, 1978 (to be published).

[4]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965).

[5]C. Fan and F. Y. Wu, Phys. Rev. B 2, 723 (1970).

[6]L. A. Takhtadzhan and L. D. Faddeev, Usp. Math. Nauk 34, 13 (1979).

[7]F. R. Gantmacher, *Applications of the Theory of Matrices* (Interscience, New York, 1959), pp. 64, 101, 107, and 111.

[8]R. Baxter, in *Fundamental Problems in Statistical Mechanics*, edited by E. G. D. Cohen (North-Holland, Amsterdam, 1980), Vol. V.

[9]Yu. G. Stroganov, Phys. Lett. 74A, 116 (1979).

[10]C. L. Schultz, Phys. Rev. Lett. 46, 629 (1981); J. H. H. Perk and C. L. Schultz, State University of New York at Stony Brook Report No. ITP-SB-81.25, to be published.

[11]A. B. Zamolodchikov and V. A. Fateev, Yad. Fiz. 32, 581 (1980) [Sov. J. Nucl. Phys. 32, 298 (1981)].