Subharmonic Sequences in the Faraday Experiment: Departures from Period Doubling

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Subharmonic sequences in the shallow-water surface waves generated in a resonator with one-dimensional properties are shown to exhibit departures from period doubling for long-period subharmonics.

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Recent interest in driven nonlinear systems has focused on the transition from the preturbulent to the "weakly turbulent" regimes. It is thought that such transitions in deterministic systems with limited degrees of freedom might provide insight into turbulence in fluids.[1]

While it has generally been accepted[2] that nonlinearities can lead to higher harmonics (as well as sum and difference frequencies), it has recently been reemphasized that driven nonlinear systems can also generate *sub*harmonics. Feigenbaum[3] has shown that a universal period-doubling sequence arises quite generally in nonlinear one-dimensional maps, appearing as a precursor to chaotic behavior. Recent computer experiments on the driven anharmonic oscillator[4] and Rössler attractor[5] exhibit this universal period-doubling sequence. The early work of Pedersen[6] on strongly driven loudspeakers at frequency f clearly evinces the beginning of a period-doubling sequence ($f/2$, $f/4$).[7] Longer sequences have also been observed in dc-driven Rayleigh-Bénard convective instabilities.[8-10]

Rayleigh[11] already noted that two classes of driven systems possess subharmonic response: driven nonlinear oscillators[12] (described by a Duffing equation, where the drive appears as an external time-varying force) and parametric excitation[13] (described by a Mathieu equation, where the drive appears as a time variation of one of the parameters). The first observation, 150 years ago, of subharmonic parametric excitation is due to Faraday.[14] He studied shallow-water waves when the containing vessel was driven vertically, and observed an $f/2$ response. Rayleigh[11] analyzed and experimentally confirmed these results.[15]

An alternate route to turbulence[16] is followed in some Rayleigh-Bénard[17,18] and couette-flow[19] experiments with the appearance of incommensurate freqencies. The convective instabilities inherently invove more than one degree of freedom, which could account for the presence of these frequencies. In order to isolate the effects of period multiplication, it is important to inhibit this other mechanism. Our version of the Faraday experiment therefore utilizes a resonator with response along only one of its three dimensions. We find a rich spectrum of subharmonic instabilities, which, however, does not generally follow universal period-doubling sequences for long-period subharmonics.[20]

Our resonator is a narrow Plexiglas annulus (Fig. 1) which is, to good approximation, a one-dimensional wave guide closing on itself. The anharmonicity of the modes caused by the curvature is negligible. The annulus has the advantage of allowing arbitrary spatial phase for the various harmonics. However, this makes determina-

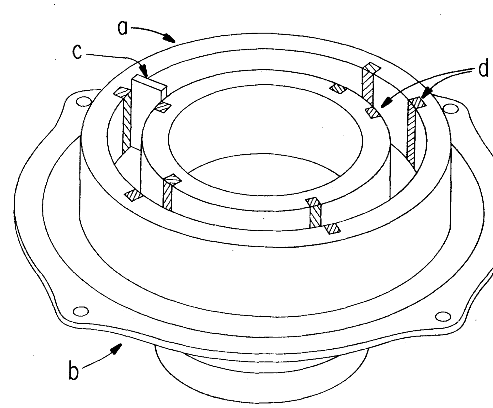

FIG. 1. Sketch of the annular plexiglas resonator, a, sitting on top of a loudspeaker, b. The resonator can be used with or without the rigid reflector, c. The wave height of the salt water in the annulus is determined by measuring the conductance between the copper blocks, d. Annulus average radius is 4.8 cm and the cross section is 0.8×2.5 cm.

tion of the amplitudes more difficult since it is unknown where the antinodes are relative to the transducers (copper-block pairs). For this reason five transducers were installed at various angles. We can also place a rigid reflector at a transducer, causing an antinode in the height oscillation to occur there. Salt and a wetting agent were added to the water. The conductance between the copper blocks gives a measure of the height oscillation.

When we oscillate the resonator with an acceleration $\sim g/3$, large-amplitude standing waves form, with peaks as high as 20 mm and troughs as low as 1 mm. By taking the power spectrum of the height we find that period multiplication occurs but not necessarily by a factor of 2. In approximately 100 cases we find that the allowed subharmonic series are given by pf/m, where p and m are integers and (with two exceptions to be discussed later) m is even. This is unlike the Rayleigh-Bénard period-doubling sequences,[8-10] where m is restricted to powers of 2.

A typical preturbulent series, $m = 14$, is shown in Fig. 2. The presence of $f/14$ implies that the water does not repeat its motion until fourteen cycles of the drive have passed. There is no ambiguity in identifying the subharmonic since *all* of its harmonics are present to beyond $2f$, many at least 40 dB above the base line, and there are no unaccounted-for peaks.

The values of m that we have observed so far at various drive amplitudes and frequencies without the rigid reflector are

$$m = 1, 2, 4, 12, 14, 16, 18, 20, 22, 24, 28, 35$$

and with the rigid reflector in place they are

$$m = 1, 2, 3, 4, 6, 12, 16, 18, 24, 26, 28, 30, 32, 34.$$

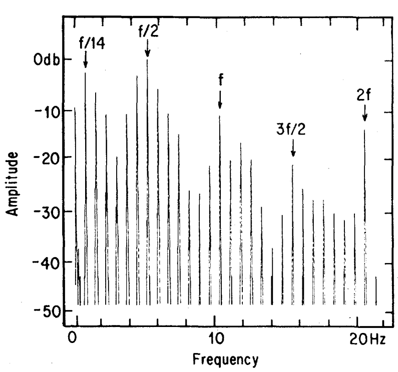

FIG. 2. Power spectrum when the resonator is oscillated at f and response is seen at $f/14$ and its harmonics.

We are optimistic that with additional effort the list will grow. Note that all the prime numbers from 1 to 17 are present as factors of m in this list.

If we fix the drive frequency and carefully increase the drive amplitude we see sequences of period multiplications which always (except for $f/3$) start with at least one factor of 2 but later develop larger factors. We have seen, for example, the sequence $m = 1, 2, 4, 24$. The sequence is reversible, with some hysteresis, if the drive is lowered. The jump from $m = 4$ to $m = 24$ is by a factor of 6 without intermediate factors of 2 and 3. The sequence can also drop and pick up factors; once, on the same run, we observed a jump from $m = 24$ to $m = 28$.

When we pick the drive frequency to be near the mth, low-amplitude mode of the resonator we tend to see the mth subharmonic series at large amplitude. The pf/m member of the series then occurs approximately at the pth resonance of the annulus. The ability to make every frequency in a subharmonic series approximately coincide with a mode regardless of m value is a great strength of our resonator. By way of contrast, in a preliminary experiment using a Petri dish we saw only $m = 2$ and a weak $m = 4$ before turbulence set in. In a cylinder there are no low-lying harmonic overtones.

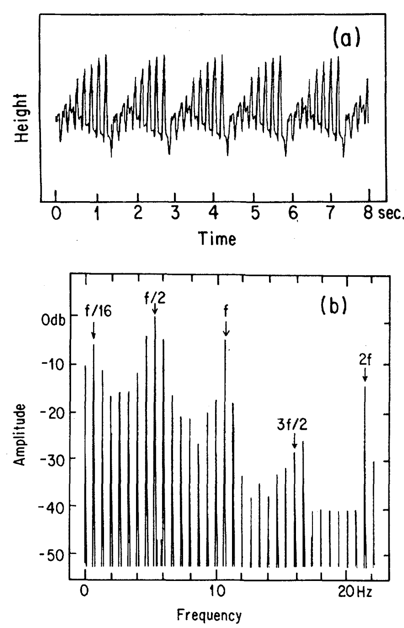

FIG. 3. (a) Height vs time of the $f/16$ periodic (preturbulent) state. (b) Power spectrum of (a).

It is interesting to compare the response in frequency and time of the periodic (preturbulent) and near-periodic (weakly turbulent) states. Figure 3(a) is a wave height versus time plot in the periodic state. The amplitude builds up and suddenly dies in a characteristic "cockscomb" pattern in which even minute details repeat. The state is robust; we have seen it persist for hours, and it is stable to small drive-amplitude changes. The periodicity is reflected in the power spectrum [Fig. 3(b)] by the absence of a visible noise floor. The spectrum also shows that this is an $m = 16$ subharmonic, consistent with the fact that there are eight cycles of $f/2$ per cockscomb. If we raise the drive amplitude by 28% we see an example of the near-periodic state (Fig. 4). Cockscombs still tend to form but end randomly, and the spectrum shows broadband noise along with periodicity at $m = 32$. In the list of observed m values, 2 and 4 were always observed in the periodic state, but the remaining m-value observations were about equally divided between the periodic and quasiperiodic states.

Earlier we noted two exceptions to the even-m series. We have seen a $pf/3$ and a $pf/35$ series in the near-periodic modes. They are the only odd series we have observed although it must be said that no concerted effort in this direction was made. The latter series is unmistakably $pf/35$ rather than $pf/34$ or $pf/36$ since unlike all other spectra, $f/2$ is not only not dominant, it is in the noise and unobservable. It is instinctive to reject odd-m modes as a possibility. However, this must not be dismissed given the observed departure from period doubling. Sum and difference frequencies are the hallmark of nonlinear processes, and the improbable presence of simultaneous excitation of two of these sequences can give such odd modes. For example, $f/10 - f/14 = f/35$.

We hope in the near future to use superfluid helium as the fluid in this experiment.

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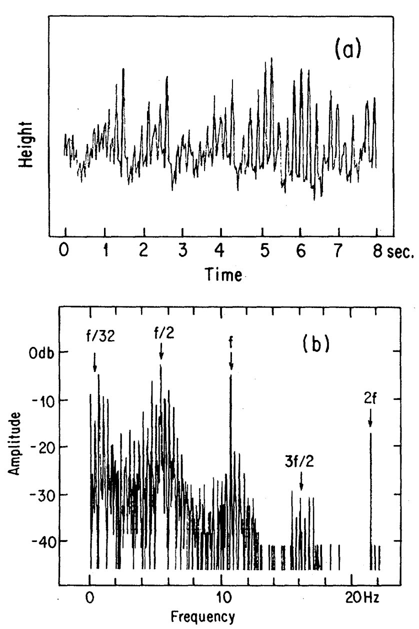

FIG. 4. (a) Height vs time of the $f/32$ quasiperiodic (weakly turbulent) state when the drive amplitude was increased 28% over that used in Fig. 3. (b) Power spectrum of (a). The 0-dB level coincides in Figs. 3 and 4.

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