form way, starting from an arbitrary classical Lie group, analytically solvable lattice models, among which is a far-reaching generalization of the XXX model, given by a quantum Hamiltonian H_{N-2}". The proposed S-matrix method will be instrumental in finding energy levels, and in the computation of correlation functions for the known and new lattice models of Toda and XXX type considered here.

[1]E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge Univ. Press, Cambridge, 1927).

[2]P.-A. Vuillermont and M. V. Romerio, J. Phys. C 6, 2922 (1973); L. Onsager, Phys. Rev. 65, 117 (1944); J. M. Drouffe and C. Itzykson, Phys. Rep. 38C, 133 (1975).

[3]R. Baxter, Ann. Phys. (N.Y.) 76, 1, 25, 48 (1973).

[4]Here $G[[\theta]]$ denotes the formal power series in θ with the coefficients from G.

[5]A. B. Zamolodchikov, Commun. Math. Phys. 69, 165 (1979); E. K. Sklanin, L. A. Takhtadjan, and L. D. Faddeev, Theor. Math. Phys. (USSR) 40, 688 (1980); L. Takhtadjan and L. D. Faddeev, Usp. Mat. Nauk 34, 13 (1979).

[6a]C. L. Schultz, Phys. Rev. Lett. 46, 629 (1981).

[6b]D. V. Chudnovsky and G. V. Chudnovsky, Phys. Lett. 98B, 83 (1981), and 79A, 36 (1980).

[7]D. V. Chudnovsky and G. V. Chudnovsky, Lett. Math. Phys. 5, 43 (1981).

[8]R. J. Baxter, Trans. Royal Soc. London, A289, 315 (1978), and to be published.

[9]A. A. Kirillov, *Representation Theory* (Springer, Berlin, 1972); R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications* (Wiley, New York, 1974).

[10]N. Bourbaki, *Groupes et Algèbres de Lie* (Herman, Paris, 1968).

[11]M. Gutzwiller, Ann. Phys. (N.Y.) 124, 347 (1980), and to be published.

[12]K. M. Case, J. Math. Phys. 15, 2166 (1974); K. M. Case and M. Kac, J. Math. Phys. 14, 594 (1973).

Linearization of the Korteweg–de Vries and Painlevé II Equations

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A new integral equation which linearizes the Korteweg-de Vries and Painlevé II equations, and is related to the potentials of the Schrödinger eigenvalue problem, is presented. This equation allows one to capture a far larger class of solutions than the Gel'fand-Levitan equation, which may be recovered as a special case. As an application this equation, with the aid of the classical theory of singular integral equations, yields a three-parameter family of solutions to the self-similar reduction of Korteweg -de Vries which is related to Painlevé II.

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Since the work of Gardner *et al.* in 1967,[1] there has been wide interest in the analysis of nonlinear evolution equations solvable by the so-called inverse-scattering transform (IST). The prototype example is the Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

In this note we shall present a new linear integral equation which, in principle, allows one to capture a far larger class of solutions than does the Gel'fand-Levitan equation. Specifically we claim that if $\varphi(k; x, t)$ solves

$$\varphi(k; x, t) + i\exp[i(kx+k^3t)]\int_L \frac{\varphi(l; x, t)}{l+k}\,d\lambda(l) = \exp[i(kx+k^3t)], \tag{2}$$

where $d\lambda(k)$ and L are an appropriate measure and contour, respectively, then

$$u = -\frac{\partial}{\partial x}\int_L \varphi(k; x, t)\,d\lambda(k) \tag{3}$$

solves the KdV equation. The well-documented physical significance of the KdV equation, of its self-similar analogue, and of the associated Schrödinger scattering problem require us to attempt to characterize the form of the most general solution/potential possible.

We now enumerate the basic results given in this note. (i) We give a direct proof that (2) and (3) solve (1); (ii) we show how the well known Gel'fand-Levitan equation can be obtained from (2) as a special case; and (iii) we characterize by a matrix Fredholm equation a three-parameter family of solutions to the similarity ordinary differential equation of (1) which is directly related to the classical second equation of Painlevé (P II). We end with some remarks regarding the role of Bäcklund transformations and relevant generalizations.

We now consider (i). The point of view we take here is, in spirit, similar to that of Zakharov and Shabat.[2] Specifically, by direct calculation we show that solutions of (2) substituted in (3) satisfy (1). We make two assumptions: (a) $d\lambda$ and L are such that differentiation by x, t may be interchanged with $\int_L$; (b) the homogeneous integral equation has only the zero solution. Defining $\tilde{L}=\tilde{L}_0+3u\partial_x$, where $\tilde{L}_0=\partial_t+\partial_x{}^3$, after some manipulation we find

$$\tilde{L}\varphi(k;x,t)+i\exp[i(kx+k^3t)]\int_L \frac{\tilde{L}\varphi(l;x,t)}{l+k}d\lambda(l)=3k[k\varphi_x+i\varphi_{xx}+iu\varphi]. \tag{4}$$

Similar calculations show that the quantity in brackets in the right-hand side of (4) satisfies the homogeneous integral equation. Hence $\tilde{M}\varphi=k\varphi_x+i\varphi_{xx}+iu\varphi=0$ which implies $\tilde{L}\varphi=0$, whereupon $\partial_x\int_L(\tilde{L}\varphi)d\lambda=0$ is (1). Moreover the equation $\tilde{M}\varphi=0$ is directly related to the Schrödinger eigenvalue problem. If we define

$$\varphi(k;x,t)=\psi(k;x,t)\exp[i(kx+k^3t)/2],$$

then $\tilde{M}\varphi=0$ gives

$$\psi_{xx}+(\tfrac{1}{2}k)^2\psi+u\psi=0. \tag{5}$$

Next we pass on to (ii). The classical theory of inverse scattering and appropriately decaying solutions of KdV may be most easily obtained as follows. Let the measure $d\lambda(k)=r_0(\frac{1}{2}k)dk/2\pi$, where $r_0(k)$ is the usual reflection coefficient of $u(x,0)$ and the contour L goes over all the poles of $r_0(k)$. [Here we have assumed, for convenience, that $u(x,0)\to 0$ rapidly as $|x|\to\infty$.] Then substituting the expression for φ into (2), defining

$$K(x,y,t)=-(\tfrac{1}{2})\int_L\psi(k;x,t)\exp[i(ky+k^3t)/2]d\lambda(k),$$

and using

$$\exp[i(k+l)x/2]/(l+k)=-i\{\int_x^2\exp[i(k+l)\xi/2]/2d\xi\}$$

(k, l satisfy $\mathrm{Im}k, \mathrm{Im}l>0$), we obtain

$$K(x,y;t)+F(x+y;t)+\int_x^\infty K(x,\xi;t)F(\xi+y;t)d\xi=0, \tag{6}$$

where

$$F(x,t)=(\tfrac{1}{2})\int_L\exp[i(kx/2+k^3t)]d\lambda(k),$$

and $u(x,t)=2\partial_xK(x,x;t)$. Hence by choosing the above measure $d\lambda$ and contour L, the Gel'fand-Levitan equation (6) may now be completely bypassed.

Soliton solutions of (1) may be calculated in a particularly easy manner from (2). Locations of the poles on the imaginary k axis in $r(k,0)$ correspond to soliton amplitudes, and the residues of $r(k,0)$ at these locations play the role of the normalization coefficients. Pure solitons may also be obtained by taking the measure as

$$d\lambda(k)_s=\sum_{j=1}^N c_j\delta(k-i\kappa_j)dk$$

(L passes through the $k=i\kappa_j$). Then (2) reduces to a linear *algebraic* system from which the well known N-soliton solution is immediately obtained.

We now discuss (iii). The KdV equation admits the similarity transformation $u(x,t)=U(x')/(3t)^{2/3}$, where $x'=x/(3t)^{1/3}$. The equation for U is given by (dropping the primes)

$$K_1(U)=U'''+6UU'-(2U+xU')=0. \tag{7}$$

We note that (7) is directly related to P II:

$$P_2(V)=V''-xV-2V^3=\alpha. \tag{8}$$

Specifically we note that the transformations $U=-V^2-V'$, $V=(U'+\alpha)/(2U-x)$ relate (8) to the equation

$$K_2(U)=U''+2U^2-xU+[\nu+U'-(U')^2]/(2U-x)=0$$

with $\nu=\alpha(\alpha+1)$. However, by direct calculation $[(2U-x)K_2(U)]'=(2U-x)K_1(U)$, hence $K_2(U)$ is an integral of (7), and thus there is a direct transformation between (7) and (8).[3] One may make use of these transformations fo find all the known (see, for example, Lukashevich[4] and Erugin[4]) elementary solutions of P II. Ablowitz and Segur[5] had established a connection between P II and IST and had characterized a one-parameter family of solutions via the Gel'fand-Levitan equation. Recently Flaschka and Newell[6] considered P II via monodromy theory. In the latter work the authors derived a formal system of linear singular integral equations for the general solution of P II. However, the highly nontrivial question of existence of solutions was left open.

An application of the result presented above in (i) is that a three-parameter family of solutions of (7) may be obtained from the linear singular integral equation

$$\varphi(t)+\frac{b(t)}{i\pi}\int_L\frac{\varphi(\tau)}{\tau+t}d\tau=f(t),\quad t\text{ on }L, \tag{9}$$

where $b(t)=f(t)=\exp[i(tx+t^3/3)]$ and $\int_L=\sum_{j=1}^5\hat\rho_j\int_{L_j}$ (see Fig. 1), $\hat\rho_1=\hat\rho_2=\rho_1$, $-\hat\rho_3=\hat\rho_4=\rho_2$, $\hat\rho_5=\rho_3$. (Hereafter j always stands for $j=1,\ldots,5$). The solution to (7) is then obtained from

$$U=\frac{1}{\pi}\frac{\partial}{\partial x}\int_L\varphi(\tau)d\tau$$

(φ depends parametrically on x). We note that both (9) and U are obtained from (2) and (3) by a

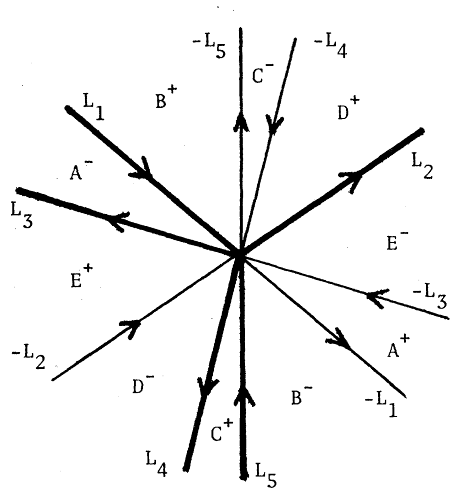

FIG. 1. Contours associated with Eqs. (9) and (12).

self-similar reduction. Moreover the contours L_j are obtained by finding the solution to the linear problem $(U=w_x)$ $w'''-(w+xw')=0$ in terms of integral representations and then deforming these contours so that they all pass through the origin. For example, note that L_1+L_2 may be deformed to the usual Airy-function contour. If we restrict ourselves to this Airy contour, the result in Ref. 5 is obtained in the same manner as that in (ii) above.

We shall proceed to demonstrate that (9) may be reduced to a system of Riemann-Hilbert problems which are solvable using Fredholm theory. For this we need the full power of the classical theory of singular integral equations.[7-9]

Consider the sectionally holomorphic function

$$\Phi(z)=\frac{1}{2\pi i}\int_L\frac{\varphi(\tau)}{\tau-z}d\tau. \tag{10}$$

The lines of discontinuity of $\Phi(z)$ are L_j; thus using the Plemelj formulas, we have

$$\begin{aligned}\Phi^+(t)=\Phi^-(t)&=\frac{1}{2\pi i}\int_L\frac{\varphi(\tau)}{\tau-t}d\tau,\quad t\text{ on }-L_j,\\ \Phi^\pm(t)&=\pm\tfrac{1}{2}\hat\rho_j\varphi(t)+\frac{1}{2i\pi}\int_L\frac{\varphi(\tau)}{\tau-t}d\tau,\quad t\text{ on }L_j,\end{aligned} \tag{11}$$

where $\Phi^\pm(t)$ for t on L_j has the standard definitions[7-9] of limits of $\Phi(z)$ as $z\to t$ from the "left-hand side" (+) and "right-hand side" (−) of L_j, and where principal-value integrals are implied when needed. With use of (11), and Eq. (9) for t on L_j and $-t$ on $-L_j$, we obtain a system which we choose to write in the form

$$\underline{\Phi}^+(t)=G(t)\underline{\Phi}^-(t)+\underline{F}(t),\quad t\text{ on }\overline{L}_j, \tag{12}$$

where $\overline{L}_j=L_j+(-L_j)$, $\underline{\Phi}^+(t)=[\Phi^+(t),\Phi^-(-t)]^T$, $\underline{\Phi}^-(t)=\underline{\Phi}^+(-t)$, $\underline{F}(t)=[f(t)H(t),-f(-t)H(-t)]^T$, $H(t)=\{\hat\rho_j$ if t on L_j, 0 if t on $-L_j\}$ and the components of the 2×2 matrix $G(t)$ are $G_{11}(t)=-2b(t)\times H(t)=-G_{22}(-t)$, $G_{12}=G_{21}=1$.

One can prove the following statements.

(a) $\Phi^+(-t)$, $\Phi^-(-t)$ are "minus" and "plus" functions, respectively. (b) Necessary conditions for solvability of (12) are the symmetry conditions $G(t)=[G(-t)]^{-1}$, $\underline{F}(t)+G(t)\underline{F}(-t)=0$, which are satisfied by the above $G,\underline{F}$. (c) Thus (12) defines a system of discontinuous Riemann-Hilbert problems with the additional restriction that $\underline{\Phi}^-(t)=\underline{\Phi}^+(-t)$. However this condition can be relaxed since one can show that (12) always admits a solution with this restriction, and moreover, in our case the solution is unique.

In order to solve (12) we first consider the homogeneous problem. The standard procedure is to transform the discontinuous homogeneous problem to a continuous one, and then obtain the fundamental set of solutions.

Associated with a given contour $\bar{L}_j$, define the following auxilliary functions

$$\omega_{jk}^{+}(t)=\left(\frac{t}{t+z_j}\right)^{\lambda_{jk}},\quad \omega_{jk}^{-}(t)=\left(\frac{t}{t-z_j}\right)^{\lambda_{jk}},$$
$$\omega_{jk}=\frac{\omega_{jk}^{+}}{\omega_{jk}^{-}},\ k=1,2, \tag{13}$$

where z_j is some j-dependent fixed point off $\bar{L}_j$. The branches of the above functions are chosen such that ω_{jk}^{+} and ω_{jk}^{-} are plus and minus functions, respectively (e.g., the branch cut for ω_{jk}^{+} is taken between 0, $-z_j$ and hence lies to the right of $\bar{L}_j$). The properties $\omega_{jk}(0+)=\exp[-i\pi\lambda_{jk}]$, $\omega_{jk}(0-)=\exp[i\pi\lambda_{jk}]$, $\omega_{jk}^{+}(t)=\omega_{jk}^{-}(-t)$ allow us to map the homogeneous system $\underline{\Phi}^{+}(t)=G(t)\underline{\Phi}^{-}(t)$ which has a discontinuity at $t=0$ to the following Riemann-Hilbert system which is continuous at the origin:

$$\underline{\Psi}^{+}(t)=g(t)\underline{\Psi}^{-}(t),\ t\text{ on }\bar{L}_j, \tag{14}$$

where we have used the transformation $\underline{\Phi}^{+}(t)=A\Omega^{+}(t)\underline{\Psi}^{+}(t)$, $\underline{\Phi}^{-}(t)=A\Omega^{-}(t)\underline{\Psi}^{-}(t)$ and hence $g(t)=[\Omega^{+}(t)]^{-1}A^{-1}G(t)A\Omega^{-}(t)$, with A, $\Omega^{\pm}(t)$ defined by

$$A_j=\begin{pmatrix}\frac{1-\Lambda_{j2}}{2\hat{\rho}_j}\exp[i\pi\lambda_{j1}/2] & \frac{1-\Lambda_{j1}}{2\hat{\rho}_j}\exp[i\pi\lambda_{j2}/2]\\ \exp[i\pi\lambda_{j1}/2] & \exp[i\pi\lambda_{j2}/2]\end{pmatrix},\quad \Omega_j^{\pm}=\begin{pmatrix}\omega_{j\alpha}^{\pm} & 0\\ 0 & \omega_{j\beta}^{\pm}\end{pmatrix}, \tag{15}$$

where for $j=2,3,4$ we have $\alpha=1$, $\beta=2$ and for $j=1,5$ we have $\alpha=2$, $\beta=1$; the λ_{jk} and Λ_{jk} are defined by

$$\exp[i\pi\lambda_{j1}]=\hat{\rho}_j+(1+\hat{\rho}_j^{2})^{1/2},\quad \exp[i\pi\lambda_{j2}]=-\hat{\rho}_j+(1+\hat{\rho}_j^{2})^{1/2},\quad \Lambda_{j1}=\exp[2i\pi\lambda_{j1}],\quad \Lambda_{j2}=\exp[2i\pi\lambda_{j2}].$$

The matrix $g(t)$ has the properties $g(t)=[g(-t)]^{-1}$ and $\det g=-1$.

One may characterize a solution of the system (14) by imposing the condition $\underline{\Psi}(z)\to\underline{\Psi}_A^{-}=\underline{\gamma}$ as $|z|\to\infty$ in A^{-}. This leads to a Fredholm equation for, say, $\underline{\Psi}^{+}(t)$, which however must be interpreted in a suitable principal-value sense as it does not converge in the normal sense at infinity. Alternatively, one may obtain a regular Fredholm equation of the second kind by imposing conditions at a finite point off all contours, say $z=1$. This leads to the following Fredholm equation for $\underline{\Psi}^{+}(t)$:

$$\underline{\Psi}^{+}(t)+\frac{1}{2\pi i}\int_{\bar{L}}\left[\frac{1}{\tau-t}-\frac{1}{\tau-1}\right][g_j(t)g(-\tau)-I]\underline{\Psi}^{+}(\tau)d\tau=g_j(t)\underline{\beta},\quad t\text{ on }\bar{L}_j, \tag{16}$$

where $\underline{\beta}=\underline{\Psi}(1)$, $\int_{\bar{L}}=\sum_{j=1}^{5}(\int_{L_j}+\int_{-L_j})$, and I is the unit matrix. Any two linearly independent $\underline{\beta}$ vectors, say $\underline{\beta}_{1,2}$, lead to a fundamental matrix $Y^{+}(t)=[\underline{\Psi}_1^{+}(t),\underline{\Psi}^{+}(t)]$ for the system (14).

With use of the above results the fundamental matrix of the discontinuous problem (12) is given by

$$X^{+}(t)=A\Omega^{+}(t)[\underline{\Psi}_1^{+}(t),\underline{\Psi}_2^{+}(t)]. \tag{17}$$

Hence the solution of (12) is given by

$$\underline{\Phi}^{+}(t)=\frac{\underline{F}(t)}{2}+\frac{1}{2\pi i}X^{+}(t)\int_{\bar{L}}\frac{[X^{+}(\tau)]^{-1}\underline{F}(\tau)}{\tau-t}d\tau. \tag{18}$$

Having obtained $\underline{\Phi}^{+}(t)$ and using (11) to obtain $\varphi(t)$, we have characterized a three-parameter family of solutions of U. With use of the results of Fredholm's theory the nonmovable critical-point property of U is easily verified.

Finally, we make some remarks. First, we only expect from (2) to obtain solutions to PII in the range $-\frac{1}{2}<\alpha<\frac{1}{2}$. To obtain the solution for all ranges of α, we believe, the Bäcklund transformations (following Rosales[10]) and "finite perturbations" (see, for example, Ablowitz and Cornille[11]) of suitable elementary solutions must be employed. Similarly, wider classes of solutions to KdV should be obtainable this way (we shall remark on this more completely in the future). Second, straightforward generalizations to the higher-order KdV equations, as well as to many other nonlinear evolution equations, are possible. Third, motivation for some of the ideas in this note originate from the concept of summing perturbation series. Relevant perturbation series can be readily developed (see, for example, Refs. 11 and 12).

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[1]C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. 19, 1095 (1967).

[2]V. E. Zakharov and A. B. Shabat, Funktsional. Anal. Prilozhen. 8, 43 (1974) [Functional Anal. Appl. 8, 226 (1974)].

[3]A. Fokas and M. J. Ablowitz, to be published.

[4]N. A. Lukashevich, Differ. Uravn. 7, 1124 (1971) [Differ. Equat. 7, 853 (1971)]; N. P. Erugin, Differ. Uravn. 12, 579 (1976) [Differ. Equat. 12, 405 (1976)].

[5]M. J. Ablowitz and H. Segur, Phys. Rev. Lett. 38, 1103 (1977).

[6]H. Flaschka and A. C. Newell, Commun. Math. Phys. 76, 65 (1980).

[7]N. I. Muskhelishvili, in *Singular Integral Equations*, edited by J. Radok (Noordhoff, Groningen, 1953).

[8]F. D. Gakhov, in *Boundary Value Problems*, edited by I. N. Sneddon (Pergamon, New York, 1966), Vol. 85.

[9]N. P. Vekua, in *Systems of Singular Integral Equations*, edited by J. H. Ferziger (Gordon and Breach, New York, 1967).

[10]R. Rosales, Stud. Appl. Math. 59, 117 (1978).

[11]M. J. Ablowitz and H. Cornille, Phys. Lett. 72A, 277 (1979).

[12]S. Oishi, J. Phys. Soc. Jpn. 47, 1037 (1979).