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Proof of the Triviality of φ_d^4 Field Theory and Some Mean-Field Features of Ising Models for $d > 4$

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It is rigorously proved that the continuum limits of Euclidean φ_d^4 lattice fields are free fields in $d > 4$. An exact geometric characterization of criticality in Ising models is introduced, and used to prove other mean-field features for $d > 4$ and hyperscaling in $d = 2$.

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(1) *The main result.*—A constructive approach to the Euclidean φ_d^4 field (in R^d) is to define it as a continuum limit of lattice fields, with the distribution

$$\prod_x (d\varphi_x) \exp[-\sum_x(\lambda_0\varphi_x^4 + B_0\varphi_x^2) + \sum_{|x-y|=1} \tfrac{1}{2}J\varphi_x\varphi_y]/\text{norm}. \tag{1}$$

The field's "Schwinger functions" are constructed as rescaled correlation functions

$$S_n^{(\text{continuum})}(x_1,\ldots,x_n) = \lim_{(\eta\to\infty)} \alpha^n\langle\varphi_{x_1\eta}\cdots\varphi_{x_n\eta}\rangle^{(\text{lattice})}. \tag{2}$$

In our notation the (cubic-) lattice spacing is fixed as 1. On the scale of the continuum the spacing is $1/\eta$. The bare parameters λ_0, B_0, J, and α are varied (renormalized), in the one-phase region, to ensure that the two-point function converges, as a density of a measure, to a locally integrable limit. [Since $S_2(\cdots) \geq 0$, local integrability means that for every finite $R \int_{|x|<R} S_2(x)\,dx < \infty$.]

This Letter is a report on some rigorous results, which answer the question whether the above procedure can lead, in high dimensions, to a field with action of order higher than 2, that is, one which is not Gaussian (i.e., a "generalized free field"). As I show,[1] in more than four dimensions the answer is *negative* [as opposed to the "superrenormalizable" cases $d = 2,3$ (Ref. 2)].

The proof is by a *nonperturbative* analysis which is based on a new (exact) *geometric* characterization of critical phenomena in the Ising model and related systems, one of which is the φ^4 field. In these systems the fields, or spins, are correlated by means of associated *currents*, in a representation which is similar to one which has been used to generate "high-temperature expansions." The "triviality" of the continuum field theory is presented as a consequence of the fact that in more than four dimensions random currents miss each other, resembling, in this respect, *Brownian paths*. (This intuition is somewhat related to ideas of Symanzik,[3] of which I was made aware by T. Spencer.) The concrete analysis of these systems is simple because of new *correlation inequalities* which are derived with use of probabilistic-geometric arguments.

The φ_d^4 field is closely related to the ferromagnetic Ising model, which consists of spin variables $\sigma_x = \pm 1$, $x \in Z^d$, with the equilibrium proba-

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bility distribution proportional to

$$\exp[\beta^{1/2}\sum_{x,y} J_{x,y}\sigma_x\sigma_y] \equiv \exp[-\beta H(\sigma)] \quad (3)$$

(with $J_{x,y} \geq 0$ of finite range, and zero magnetic field). For low values of the inverse temperature β, the correlation $\langle\sigma_x\sigma_y\rangle$ has an exponential decay characterized by the "mass gap," defined by maximizing the values of m for which $\langle\sigma_x\sigma_y\rangle \leq \text{const}\exp(-m|x-y|)$. For $d \geq 2$ there is a critical $\beta_c < \infty$, such that the correlation length $\xi \equiv m^{-1}$ diverges as $\beta \nearrow \beta_c$. Near the critical point, one obtains interesting continuum limits with use of an analog of (2), e.g., with

$$\eta = \xi(\beta), \quad \alpha = \langle\sigma_0\sigma_\xi\rangle^{-1/2}. \quad (4)$$

For the nearest-neighbor interaction (and, generally, systems for which an "infrared bound" is obeyed) I show that the last limit is Gaussian, if $d > 6$. The proof for dimensions $4 < d \leq 6$ is reduced to the verification that $\langle\sigma_0\sigma_\xi\rangle \geq \text{const}|\xi|^{-(d-2)}$—a generally conjectured property of the two-point function (i.e., the critical exponent $\eta = 0$) for $d > 4$. The reason that the implications seem stronger for the φ^4 field theory than for the Ising model is that in the former case one considers only limits in which S_2 is locally integrable. While regularity at short distance has not been proven for the special limit (4) this limit is certainly natural from the point of view of statistical mechanics. Other results for $d > 4$, and one for $d = 2$, are described at the end of this Letter.

(2) *An exact relation of long-range order to percolation of currents.*—Referring to pairs of sites as bonds, $b \equiv (x,y)$, the partition function for the Ising models described by (3), in a finite region Λ, is

$$Z_\Lambda = 2^{-|\Lambda|}\sum_{\sigma_x = \pm 1}\exp\Big(\sum_{b=(x,y)}\beta J_b\sigma_x\sigma_y\Big).$$

Expanding, for each bond

$$\exp(\beta J_b\sigma_x\sigma_y) = \sum_{n_b = 0,1,2,\ldots}(\sigma_x\sigma_y)^{n_b}(\beta J_b)^{n_b}/n_b!$$

and averaging over $\{\sigma_x\}$, one arrives at the relation

$$Z_\Lambda = \sum_{\partial\underline{n} = \varnothing} w(\underline{n}). \quad (5)$$

$\underline{n} = \{n_b\}$ represents an assignment of integers, to be regarded as *fluxes*, to the various bonds.

$$\partial\underline{n} = \{x \in \Lambda \mid \prod_{b \ni x}(-1)^{n_b} = -1\}$$

is the set of *sources* (≡ sinks), modulo 2, for a given current $\underline{n}$, $\varnothing$ is the empty set, and the weights of the configurations are

$$w(\underline{n}) = \prod_b (\beta J_b)^{n_b}/n_b!.$$

Applied to the correlation functions, i.e., thermal averages of $\sigma_A = \prod_{x\in A}\sigma_x$, this procedure yields

$$\langle\sigma_A\rangle = \sum_{\partial\underline{n} = A} w(\underline{n}) \Big/ \sum_{\partial\underline{n} = \varnothing} w(\underline{n}). \quad (6)$$

Thus $-\ln\langle\sigma_A\rangle$ is a measure of the increase in the free energy, of the system of currents, due to the creation of sources at A.

For small β the predominant contribution to the sum in (5) comes from currents consisting of only "local loops," whereas each term in the numerator for $\langle\sigma_x\sigma_y\rangle$ has to contain a current linking the pair of sources $\{x,y\}$. This observation has been used to generate "high-temperature expansions." While it has been expected that the phase transition of the model is somewhat related to the formation of infinitely long currents, an exact relation of this kind was established in Ref. 1.

New geometric aspects of the model are uncovered by introducing a duplicate system of independent currents $\underline{n}_1$ and $\underline{n}_2$, each having the distribution described above. We decompose the set of sites to clusters which are connected by bonds on which $n_1 + n_2 \neq 0$, and denote by $x \approx y$ the event that x and y belong to the same cluster. Using a simple combinatorial argument (lemma 1 of Ref. 4) we prove[1] the following relations:

$$\langle\sigma_x\sigma_y\rangle = \text{Prob}\left(x \approx y \,\middle|\, \begin{matrix}\partial\underline{n}_1 = \varnothing \\ \partial\underline{n}_2 = \varnothing\end{matrix}\right), \quad (7)$$

$$\langle\sigma_x\sigma_z\rangle\langle\sigma_z\sigma_y\rangle = \langle\sigma_x\sigma_y\rangle\,\text{Prob}\left(x \approx z \,\middle|\, \begin{matrix}\partial\underline{n}_1 = \{x,y\} \\ \partial\underline{n}_2 = \varnothing\end{matrix}\right). \quad (8)$$

Both expressions refer to the (normalized) probability that the sites x and y are connected by $\underline{n}_1 + \underline{n}_2$, in the duplicate system of two independent currents, each of which has only the specified sources.

Equations (7) and (8) extend to the infinite-volume limit. A striking aspect of (7) is that it provides a generally valid *exact* identification of the onset of the *long-range order*, characterized by $\lim_{|x-y|\to\infty}\langle\sigma_x\sigma_y\rangle > 0$, as a phenomenon of *percolation* in the associated system of duplicated currents.

(3) *A representation of the φ^4 lattice model.*—The relation of the φ^4 field to the Ising model

which is used in Ref. 1 is embedded in the Simon-Griffiths[5] method of representing the single-site measure $\exp[-(\lambda_0\varphi^4+B_0\varphi^2)]d\varphi$ norm as a limiting distribution of the variable $\varphi^{(N)}=(12\lambda_0)^{-1/4}\times N^{-3/4}\sum_{\alpha=1}^{N}\sigma^{(\alpha)}$, with $N\to\infty$, where $\sigma^{(\alpha)}$ are Ising spins with the mean-field Hamiltonian

$$H=-[(2N)^{-1}-B_0(12\lambda_0N^3)^{-1/2}]\sum_{\alpha,\delta=1}^{N}\sigma^{(\alpha)}\sigma^{(\delta)}$$

(at $\beta=1$).

Thus, the φ^4 lattice field is the limit, as $N\to\infty$, of a system of Ising spins, $\{\sigma_x^{(\alpha)}\}$, $x\in Z^d$, $\alpha=1,\ldots,N$, with a ferromagnetic pair interaction of range 1 (lattice spacing) which is independent of the index α. If one regards the sites x as representing blocks of a larger lattice (with the internal coordinate α), the above system may be viewed as a "local-mean-field" approximation, and the convergence to φ^4 is somewhat reminiscent of the Landau-Ginzburg theory. The exact details are not relevant for our analysis.

In the above formalism, the correlations of the spins $\sigma_x^{(\alpha)}$ are expressed by means of the associated currents which link the "microscopic" points (x,α). As a generalization of (8) we have the following bound[1] on the probability that the currents link a source with *some* point in the "block" z:

$$\mathrm{Prob}\left((x,\alpha)\approx z \,\middle|\, \begin{matrix}\partial \underline{n}_1=\{(x,\alpha),(y,\alpha)\}\\ \partial\underline{n}_2=\varnothing\end{matrix}\right)\leq\langle\varphi_x^{(N)}\varphi_z^{(N)}\rangle\langle\varphi_z^{(N)}\varphi_y^{(N)}\rangle/[\langle\varphi_x^{(N)}\varphi_y^{(N)}\rangle\langle\varphi_z^{(N)2}\rangle_0]. \tag{9}$$

$\langle\cdots\rangle_0$ represents average with respect to the single-site measure (i.e., $J\equiv 0$).

(4) *The continuum limit.* (a) *Heuristics.*—The consideration of currents offers an intuitive explanation of the "triviality" (for $d>4$) of the scaling limits of the models considered here. First an intriguing feature of the current $\underline{n}_1+\underline{n}_2$ should be pointed out. The size of the connected cluster $C_{n_1+n_2}(0)=\{x|x\approx 0\}$ can be expressed, with use of (7), as $\langle|C_{n_1+n_2}(0)|\rangle=\sum\mathrm{Prob}(x\approx 0)=\sum\langle\sigma_0\sigma_x\rangle^2$. The "infrared bound"[6] (which holds for the nearest-neighbor interaction) implies that in more than four dimensions this quantity is uniformly bounded from above for $\beta\leq\beta_c$. Since by (7) the current $\underline{n}_1+\underline{n}_2$ does span a dense infinite cluster if $\beta>\beta_c$, we learn that *the expected size of the cluster remains uniformly bounded even at the percolation threshold*!

Consider now $\langle\sigma_{x_1}\cdots\sigma_{x_{2n}}\rangle$ for $2n$ widely separated points (the odd correlations vanish). It is expressed in (6) by a sum over currents with sources at these sites. For each such current one may organize the sources into n linked *pairs*. If the (long) linking currents intersect, the pairing is not unique. However, we have just seen that for $d>4$, *even at* $\beta=\beta_c$, the system does not favor long currents, except of course for those imposed by the separate sources. Furthermore, it may be expected, on the basis of analogy with the behavior of paths of the Brownian motion, that for $d>4$ these long currents miss each other. (The intersection properties, for which $d=4$ is the critical dimension, are better understood if one observes that the trail left behind a Brownian path has "dimension" *two*.) If the effects of distant currents factorize, we should obtain

$$\langle\sigma_{x_1}\cdots\sigma_{x_{2n}}\rangle=\sum_{\text{pairings}}\langle\sigma_{x_{i_1}}\sigma_{x_{j_1}}\rangle\cdots\langle\sigma_{x_{i_n}}\sigma_{x_{j_n}}\rangle+\text{correction}, \tag{10}$$

with a small correction due to the interaction of the long currents. Indeed we prove that for $d>4$ the correction is insignificant at large separations, leaving (10) in the form which (by Wick's theorem, and its converse) characterizes Gaussian variables.

(b) *New correlation inequality.*—For $n=2$ the correction in (10) is, by definition, the truncated correlation function (for the one-phase region, $\langle\sigma_x\rangle=0$)

$$u_4(x_1,x_2,x_3,x_4)\equiv\langle\sigma_{x_1}\cdots\sigma_{x_4}\rangle-[\langle\sigma_{x_1}\sigma_{x_2}\rangle\langle\sigma_{x_3}\sigma_{x_4}\rangle+\langle\sigma_{x_1}\sigma_{x_3}\rangle\langle\sigma_{x_2}\sigma_{x_4}\rangle+\langle\sigma_{x_1}\sigma_{x_4}\rangle\langle\sigma_{x_2}\sigma_{x_3}\rangle]. \tag{11}$$

The considerations which lead to (7) and (8) permit us to perform exact cancellations. The result is the identity[1]

$$u_4(x_1,\ldots,x_4)=-2\langle\sigma_{x_1}\sigma_{x_2}\rangle\langle\sigma_{x_3}\sigma_{x_4}\rangle\,\mathrm{Prob}\left(x_1\approx x_3\,\middle|\,\begin{matrix}\partial\underline{n}_1=\{x_1,x_2\}\\ \partial\underline{n}_2=\{x_3,x_4\}\end{matrix}\right) \tag{12}$$

(and a similar identity which is manifestly symmetric).

Equation (12) implies the Lebowitz inequality $u_4\leq 0$; however, it also leads to *lower bounds* on u_4. Using inequalities which at certain place overestimate the probability that two clusters intersect by the expected size of the intersection, we derive the following new correlation inequality for the φ^4 lattice systems and Ising models[1] (where φ_x, φ_x^2 are replaced by σ_x, 1):

$$|u_4(x_1,\ldots,x_4)| \leq 2\sum_y \langle\varphi_{x_1}\varphi_y\rangle\langle\varphi_{x_2}\varphi_y\rangle\langle\varphi_{x_3}\varphi_y\rangle\langle\varphi_{x_4}\varphi_y\rangle/\langle\varphi_y^2\rangle_0^2. \qquad (13)$$

(c) "Triviality" of φ_d^4 for $d>4$.—Using (13) and the known properties of the two-point function, $\langle\varphi_0\varphi_x\rangle$, one may show that for any scaling of the bare parameters (in the single-phase region), the limiting φ^4 field theory is inevitably a Gaussian field (i.e., a "generalized free field"), assuming the limit (2) exists and is locally integrable. (It suffices[7] to prove $u_4^{(\text{continuum})} \equiv 0$.)

The complete proof is deferred to Ref. 1. However, it is instructive to consider here the (dimensionless) renormalized coupling constant $g = |\bar{u}_4|/(\chi^2\xi^d)$, where $|\bar{u}_4| = \sum |u_4(0,x_2,x_3,x_4)|$ and $\chi = \sum\langle\varphi_0\varphi_x\rangle$. By (13): $|\bar{u}_4| \leq 2\chi^4/\langle\varphi_0^2\rangle_0^2$. This inequality allows the completion of the analysis of Glimm and Jaffe,[8] who proved a universal upper bound on g. Using the bound of Sokal $2d\beta J\langle\varphi_0^2\rangle \geq e^{-m}$, and[9] $\chi \leq (1+\xi^2)/(\text{const}\,\beta J)$, one obtains

$$g \leq \text{const}/\xi^{d-4} \xrightarrow[\xi\to\infty]{} 0 \text{ for } d>4. \qquad (14)$$

More detailed arguments also show that in the limit (2), assuming local integrability, $\alpha^4 u_4^{(\text{latt})}(x_1,\eta,\ldots,x_4\eta) \to 0$ as $O(\eta^{-(d-4)})$. (I am grateful to J. Fröhlich for a stimulating discussion of this point.)

(5) Results for Ising models.—For the nearest-neighbor Ising model, and others with similar infrared behavior, (14) implies that, indeed, hyperscaling is not valid for $d>4$ (see also Sec. 1). Another prediction of the mean-field approximation, namely the critical-exponent value $\gamma=1$, is proved for such models by showing that for $\beta<\beta_c$

$$(\epsilon\sum_x J_{0,x})^{-1}|\beta-\beta_c|^{-1} \geq \chi \geq (\sum_x J_{0,x})^{-1}|\beta-\beta_c|^{-1} \qquad (15)$$

with some $\epsilon(J)>0$, for $d>4$. The lower bound has already been derived by Glimm and Jaffe. The new upper bound holds whenever $\sum\langle\sigma_0\sigma_x\rangle^2$ is finite at T_c. The last condition played a role also in the proof of the finiteness of the specific heat at T_c+0.[10]

In the other direction, for $d=2$ we prove that for any ferromagnetic interaction, $J_{x,y}=J(|x-y|)$, of range R (with $\xi_1 \equiv \sum|x^{(1)}+x^{(2)}|\langle\sigma_0\sigma_x\rangle/\chi$),

$$g_1 \equiv |\bar{u}_4|/(\chi^2\xi_1^2) \geq 1/(2R^2). \qquad (16)$$

This lower bound is an easy consequence of the fact that in two dimensions it is quite natural for two currents to intersect. Taking the logarithm of the left side of the inequality (16), we see that the corresponding sum of the critical exponents is *exactly* zero (hyperscaling). The vanishing of the lower bound when $R\to\infty$ is consistent with the expectation that spreading the interaction should, in a certain respect, lead to the mean-field limit, without affecting the critical exponents.

(6) Other applications.—The basic approach exposed here is applicable also to other systems which may include (i) external fields of either sign (including the problem of the roughening transition), (ii) antiferromagnetic interactions, and (iii) interactions of higher order, e.g., the $Z(2)$ lattice gauge model. In the above cases one is naturally led to consider properties of random surfaces—which are not yet well understood.

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