Chaos in the Einstein Equations

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A very general cosmological solution to Einstein's equations exhibiting sensitive dependence on initial data is analyzed as a dynamical system. The metric and topologic entropies of the one-dimensional Poincaré map are determined and various results in number theory employed to calculate other invariants of the "chaotic" cosmological dynamics.

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Until only recently physicists believed that the presence of random or chaotic behavior in dynamical systems always derived from random initial data, stochastic forcing, or the excitation of a very large number of degrees of freedom. Although any of these are sufficient to generate observable chaos in a dynamical system, it is now known that none are necessary.[1] Deceptively simple recursive systems, notably iterated maps of the unit interval,[2] display behavior which, although deterministic, is so sensitive to the initial data that for all practical purposes it is unpredictable. This Letter reports some applications of these mathematical developments to general relativistic cosmologies. We shall determine nonzero metric and topologic entropies[3] for the Poincaré return map of a generic homogeneous cosmological model which exhibits a sensitive dependence on initial conditions under evolution by the Einstein equations.

The clearest example of chaotic dynamics is provided by the diagonal[4] Bianchi type-IX or "Mixmaster" universe. This was first investigated by Misner[5] although we shall employ the insights of Belinskii, Khalatnikov, and Lifshitz[6] in our formulation. The three essential Einstein equations for the evolution of the orthogonal expansion scale factors are given by[5,6]

$$(\ln a^2)'' = (b^2 - c^2)^2 - a^4, \quad \text{and cyclic permutations,} \tag{1}$$

where the prime denotes $\partial_\tau \equiv abc\,\partial_t$, with t the proper time. The system (1) generates a flow, ψ_t, in the phase space. In order to discretize the evolution, one constructs the Poincaré map[2] of intersections $\psi_t \cap \mathcal{P}$ where $\mathcal{P}$ is a $(2-d)$-dimensional hypersurface in the phase space. This sequence of intersections $u_0, u_1, \ldots, u_n$ forms the Poincaré map. Belinskii, Khalatnikov, and Lifshitz[6] have shown that the evolution of (1) is characterized by a sequence of states lying close to different Kasner space-times (closed orbits), each parametrized by some real number u. The evolution passes through successive states in which the expansion rates in orthogonal directions, p_1, p_2, and p_3, satisfy algebraic identities $\sum p_i = \sum p_i^2 = 1$. The p_i can be uniquely[7] parametrized as $p_i(u)$ for $u \in (1, \infty)$. If the initial state is characterized by an irrational number u_0, then evolution proceeds via successive small oscillations close to the Kasner states coded by $u_0 - 1$, $u_0 - 2$, ..., etc., until the integer part of u_0 is exhausted. A new cycle of small oscillations then commences in which the initial state is coded by $u_1 = 1/(u_0 - [u_0])$, where $[\cdots]$ denotes the integer part thereof. An infinite number of these oscillations occur in $(0,t)$ and the number of small oscillations, k_r, occurring within the rth cycle, is given by the rth partial quotient in the infinite continued-fraction expansion (cfe) of the arbitrary irrational $u_0 \equiv \{k_1, k_2, \ldots, k_r, \ldots\}$. The Poincaré map is therefore one dimensional and there is a sensitive dependence on initial conditions: Two Mixmaster universes beginning arbitrarily close to each other will diverge exponentially fast as they evolve. To make this notion precise we note that the return map is equivalent to the piecewise-continuous map T: $[0,1] \circlearrowleft$ given by

$$x_{n+1} = T(x_n) = x_n^{-1} - [x_n^{-1}], \quad x_n \neq 0. \tag{2}$$

Also, $T(0) \equiv 0$ and if $k_n \equiv [x_n^{-1}]$ then the cfe of x_0 is $\{k_1, k_2, \ldots, k_n, \ldots\}$. The map T possesses an infinite number of discontinuities at r^{-1}, $r \in Z^+$. In analytic form,

$$T(x) = x^{-1} - r; \quad x \in (1/(r+1),\ 1/r). \tag{3}$$

We shall be interested in the stationary properties of T^n when n is large. Although T does not preserve the Lebesgue measure λ, it does[8] preserve a measure μ_0 which is absolutely continuous with respect to λ. If μ_0 is invariant then $\mu_0(A) = \mu_0(T^{-1}A)$ for any measurable set A, and for 0

$< x_k < 1$

$$\mu_0(\theta)d\theta = \sum_{k=1}^{\infty} \mu_0(x_k)dx_k, \tag{4}$$

where

$$x_k = (\theta + k)^{-1}. \tag{5}$$

So

$$\mu_0(\theta) = (1/\ln 2)\sum_{k=1}^{\infty}(\theta+k)^{-2}\mu_0(1/(\theta+k))$$
$$= 1/(\theta+1)\ln 2 \tag{6}$$

is the unique normalized invariant measure[9] for T. The existence of μ_0 characterizes the Mixmaster universe as a measurable dynamic system. Its chaotic behavior is a property of the elements (T, μ_0). If $\pi_n(y,x)$ is the joint probability that $T^n(x_0) = x$ and $T^{n-1}(x_0) = y$, then the information loss[10] under iteration by T is

$$I(x) = -\sum_{y\in T^{-1}(x)} \pi(y,x)\log_2\pi(y,x). \tag{7}$$

The expectation of information loss over the measure μ_0 is the *metric (or K-) entropy, $h(T, \mu_0)$*. A system is chaotic[11] if $h(T,\mu) > 0$; roughly speaking, neighboring trajectories diverge like $\sim\exp(hn)$ on the average. So,

$$h(T,\mu_0) = \int_0^1 I(x)\mu_0(x)dx. \tag{8}$$

Now

$$\pi(y,x) = |T'(x)|^{-1}\mu_0(y)/\mu_0(x) \tag{9}$$

and thus

$$h(T,\mu_0) = \int_0^1 \mu_0(x)\log_2|T'(x)|dx \tag{10}$$

$$= -\frac{2}{(\ln 2)^2}\int_0^1 \frac{\ln x}{1+x}\,dx = \frac{\pi^2}{6(\ln 2)^2}. \tag{11}$$

This single number, $h(T,\mu_0) = 3.4237\ldots$, invariantly characterizes the ergodic features of the Mixmaster universe. It is a comparatively large entropy, nearly five times larger than the Lorentz strange attractor.[10] Note that the entropies of the Kasner and Bianchi type-II universes which approximate the evolution of (1) over intervals containing zero and single turning points of a, b, c, respectively,[5,6] during a cycle are zero. It is the cycle-to-cycle evolution that generates information, not the small oscillation $(u \to u-1)$ phase in which T' is unity and (10) vanishes.

In view of the sensitivity of the function $T(x)$ one must ask what the consequences of a small error in its specification would be. This might be equivalent to examining the dynamics of a slightly perturbed Mixmaster universe or a "neighboring" space-time. If we alter the Poincaré map by a constant $\epsilon > 0$ to

$$x_{n+1} \equiv T_\epsilon(x_n) = x_n^{-1} - [x_n^{-1} + \epsilon],$$
$$x \in [-\epsilon, 1-\epsilon], \tag{12}$$

then a smooth invariant measure μ_ϵ still exists which is absolutely continuous with respect to λ. The metric entropy *increases* logarithmically with ϵ until $\epsilon \sim 0.38$ and then remains constant to $\epsilon = 0.5$. For $0 \leqslant \epsilon \leqslant (3-\sqrt{5})/2$,

$$h(T_\epsilon, \mu_\epsilon) = h(T,\mu_0)\ln 2/\ln(2-\epsilon). \tag{13}$$

The study of more general entropy-increasing perturbations may offer clues to the generic behavior of the Einstein equations.[12]

It can also be shown that the map T possesses very strong statistical properties derived from the Einstein equations (1). It possesses the *weak Bernoulli* property,[13] and so cannot be finitely approximated. This implies that it is *strongly mixing*,[14] and so for any measurable sets A and B the map T^nA tends, as $n\to\infty$, to occupy the same fraction of B as it does of the whole space of possibilities. This in turn implies the weaker property of *ergodicity*, that is, the Mixmaster eventually gets arbitrarily close to all possible Kasner universes.[15] T contains no strange attractor.

Another invariant dynamical entropy exists for the Mixmaster return map T alone [rather than the pair (T,μ)]. The *topologic entropy, $H(T)$*, is a measure of the number of orbits[16] of T. An orbit is an iterative sequence $(x_0, T(x_0)\cdots T^n(x_0), \ldots)$. If we denote the maximum number of different orbits of T for $x_0 \in [0,1]$ which are separated by distance exceeding ϵ after n iterations of T as $N(\epsilon, n)$, then

$$H(T) = \lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{\log_2[N(\epsilon,n)]}{n}. \tag{14}$$

Roughly speaking, the memory of initial conditions is lost after $\sim H^{-1}$ iterations. The number of orbits of T can be calculated by using standard results in continued-fraction theory.[17] As the number, r, of partial quotients in the cfe of an irrational x_0 is increased, so the better the rational approximation for x_0 that is obtained by truncating the continued fraction after r terms. If p_r and q_r are the (relatively prime) numerator and denominator of this approximant then $q_r > 2^{(r-1)/2}$ and the number of orbits separated by a distance exceeding $\epsilon_k \equiv q_k^{-1}(q_k + q_{k-1})^{-1}$ is $q_k(q_k$

$+q_{k-1}$). Since $\epsilon_k \to 0$ as $k \to \infty$, we have, by (14),

$$H(T) = \lim_{n\to\infty} \frac{2\ln q_n}{n\ln 2}. \tag{15}$$

By a theorem of Lévy,[18] it is known that for almost every x_0 the limit of $(q_n)^{1/n}$ as $n \to \infty$ equals $\exp(\pi^2/12\ln 2)$; thus

$$H(T) = \pi^2/6(\ln 2)^2. \tag{16}$$

If $H(T)$ and $h(T, \mu)$ both exist for a $(1-d)$-dimensional map T, then $H(T)$ is the maximum metric entropy[16]

$$H(T) = \sup_{\mu}\{h(T, \mu)\}. \tag{17}$$

Since μ_0 is the unique invariant measure[19] we have equality of h and H for T. The metric entropy is also equal to the Lyapunov characteristic exponent (LCE) for T and its numerical value can be compared with those of other chaotic systems.[2,10,20]

It is also possible to calculate the LCE's for some other aspects of Mixmaster evolution. Suppose we examine the rate at which the logarithmic amplitude of the oscillations increases from cycle to cycle as $t \to 0$ ($\tau \to \infty$). If we denote the ratio of $\max\{a^2, b^2, c^2\}$ to $\min\{a^2, b^2, c^2\}$ in the rth cycle by Δ_r then the LCE is defined as Λ, where[21]

$$\ln\Delta_r = k_1^2 k_2^2 \cdots k_r^2 \ln\Delta_0 \equiv \exp(\Lambda r)\Delta_0. \tag{18}$$

Remarkably, the Khinchin theorem[17,22] proves that, for almost every u with a cfe $\{k_1, k_2, \ldots, k_n, \ldots\}$, the geometric mean (unlike the arithmetic mean) of the $\{k_i\}$ converges to a universal constant, $\mathcal{K}$, such that

$$\lim_{n\to\infty} (k_1 k_2 \cdots k_n)^{1/n} = \prod_{r=1}^{\infty} [(r+1)^2/r(r+2)]^{\ln r/\ln 2} \equiv \mathcal{K}. \tag{19}$$

The infinite product converges slowly to $\mathcal{K} = 2.685\,45\ldots$. Therefore the LCE is given by $\ln\mathcal{K}^2 = 1.975\ldots$. There exist interesting members of the set of zero measure for which (19) is false: For example, $e = \{2, 1, 2, 1, 1, 4, 1, 1, 6, \ldots\}$ has

$$\lim_{n\to\infty} (k_1(e) \cdots k_n(e))^{1/n} \propto n^{1/3}. \tag{20}$$

Mixmaster trajectories chosen close to $u_0 = e$ will be observed to diverge from it more rapidly than an exponential. Since $\ln\Delta_r \sim r^r \ln\Delta_0$, the LCE would be infinite.

What is the physical origin of the chaotic dynamics that are evidenced by the nonzero entropy of the Einstein equations, (1)? The Mixmaster universe represents the evolution of gravitational waves with sufficient degrees of freedom to allow the spatial three-curvature to be anisotropic. As the gravitational waves move they generate this three-curvature anisotropy which has a backreaction upon their motion. The curvature anisotropy is entirely general relativistic in origin, and the immediate cause of the nonzero entropy of the Poincaré map. Although not yet studied by dynamicists, general-relativistic systems are anticipated to possess more spectacular chaotic behavior than classical ones because of the unique self-interacting nonlinearity of Einstein's theory. Whereas all other physical theories merely provide equations which describe the interaction of fields or particles on some fixed and preassigned space-time geometry, general relativity is different. The motion of fields and particles actually determine the general-relativistic space-time on which they interact.

A more detailed account of these and other investigations will be published elsewhere. Other dynamical aspects of general-relativistic cosmologies will be examined; in particular, the geodesic flow on backgrounds with anisotropic negative curvature. A comparative measure of the generality of space-times can be constructed by evaluating their metric entropies.[23] Finally, the investigations reported here, together with the relation between the dynamical entropy of the geodesic flow and the space-time curvature invariants, may allow a rigorous formulation of Penrose's notion[24] of "gravitational entropy." This, in turn, may prove fruitful in furthering our understanding of intrinsically thermodynamic aspects of nonstationary gravitational fields.

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[9]This result was known to Karl Friedrich Gauss. See R. O. Kuzmin, Dok. Akad. Nauk SSSR 375A (1928); P. Lévy, Bull. Soc. Math. (France) 57, 178 (1929). Iterates of T differ from this stationary distribution by quantities $< \exp(-An)$ for large n, for some constant A.

[10]C. E. Shannon and W. Weaver, *The Mathematical Theory of Communication* (Univ. of Illinois Press, Urbana, 1962); R. S. Shaw, Ph.D. thesis, University of California, Santa Cruz, 1978 (unpublished).

[11]$h(T,\mu)$ is invariant under isomorphisms $(T,\mu) \to (\tilde{T},\tilde{\mu})$, $\tilde{T} = g \cdot T \cdot g^{-1}$, $\tilde{\mu} = \mu \cdot g^{-1}$.

[12]$\epsilon > 0$ is a very "tame" perturbation; it does not alter T' but it alters $h(T,\mu)$ only via its effect on the asymptotic measure.

[13]J. Moser, E. Philips, and S. Varadhan, *Ergodic Theory* (Courant Institute of Mathematical Sciences, New York, 1975).

[14]The "Mixmaster" derives its name from another property called "mixing" by cosmologists {Misner, Ref. 5; Belinskii, Khalatnikov, and Lifshitz, Ref. 6; A. G. Doroshkevich and I. D. Novikov, Astron. Zh. 47, 948 (1970) [Sov. Astron. 14, 763 (1971)].} Mixmaster cosmologies might allow light to circumnavigate the Universe many times and so by diffusive mixing of matter allow physical irregularities to be smoothed out (Refs. 5 and 6). This should not be confused with the technical property "strongly mixing" used here. It means, for any measurable sets A and B,

$$\lim_{n\to\infty} \mu_0(T^{-n}A\cap B) = \mu_0(A)\mu_0(B).$$

[15]Ergodicity ensures that the only invariant measure absolutely continuous with respect to μ_0 is μ_0 whereas mixing allows other such measures so long as they converge to μ_0 as $n \to \pm\infty$.

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[23]Preliminary investigations indicate that, among *vacuum* models, the only spatially homogeneous universes with nonzero entropies are Bianchi types VIII and IX. This may be related to the presence of true gravitational degrees of freedom. All velocity-dominated universes have zero entropy.

[24]R. Penrose, in *General Relativity: An Einstein Centenary*, edited by S. W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, 1979).