Scaling for External Noise at the Onset of Chaos

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The effect of external noise on the transition to chaos for maps of the interval which exhibit period-doubling bifurcations are considered. It is shown that the Liapunov characteristic exponent satisfies scaling in the vicinity of the transition. The critical exponent for noise is calculated with the use of Feigenbaum's renormalization group approach, and the scaling function for the Liapunov characteristic exponent is obtained numerically by iterating a map with additive noise.

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The notion that the transition to turbulence in fluids has universality properties similar to those of critical phenomena has been suggested by Feigenbaum[1] on the basis of the scaling behavior of mathematical models near the onset of chaos.[2] A further impetus for an analogy between the transition to chaos and critical point phase transitions was given[3] by the observation that as a control parameter r in these models increases past a critical value r_c into the chaotic regime the measure-theoretic entropy—the Liapunov characteristic exponent $\bar{\lambda}$—has an envelope curve of the form $(r-r_c)^\tau$. The universal exponent τ is given by $\tau = \ln 2/\ln\delta = 0.449\,806\,9\ldots$, where δ is the maximum eigenvalue associated with perturbations about the invariant map[1] of the interval. The transition to chaos in these models is heralded by a cascade of period-doubling bifurcations,[2] which is also of interest to an understanding of the onset of turbulence in physical systems.[4]

Motivated by the interpretation of experiments in fluids[5] and solids and by some recent numerical calculations,[6,7] we have considered theoretically the effect of added external noise on the transition to chaos in maps of the interval. The main result to be reported here is that the noise amplitude behaves as a *scaling variable* and that the dependence of the Liapunov characteristic exponent $\bar{\lambda}$ on the noise amplitude σ and $\bar{r} = (r-r_c)/r_c$ is of the scaling form

$$\bar{\lambda}(r,\sigma) = \sigma^\theta L(\bar{r}/\sigma^\gamma) \tag{1}$$

with $L(y)$ a universal function, and θ and γ universal exponents. In the limit of vanishing noise $\sigma \to 0$ we have $\bar{\lambda} \propto \bar{r}^\tau$ which implies that as $y \to \infty$, $L(y) \propto y^\tau$, and leads to the exponent relation $\theta = \gamma\tau$.

The idea that the noise plays a role parallel to that of the ordering field in a ferromagnetic transition was conjectured previously in Ref. 7. The noise exponent θ is a new critical exponent which we evaluate from an extension of Feigenbaum's scaling theory. Our result agrees with the recently observed value[7] of θ to within the limits of accuracy of the measurement. We also report on the measured form of the scaling function $L(y)$.

We start out by specifying the form of the one-dimensional map with additive noise. It is defined by the stochastic recursion relation

$$x_{K+1} = f(x_K; r) + \xi_K \sigma \tag{2}$$

with $f(x;r)$ a continuous function of x in a finite interval having a parabolic maximum, and r a parameter that controls the shape of the function.[2] A common example is the function $rx(1-x)$ with $0 \leqslant r \leqslant 4$, and $0 \leqslant x \leqslant 1$. The quantity ξ_K is a random variable controlled by an even distribution of unit width, and σ is a variable that controls the width (or amplitude) of the noise. Note that when $\sigma = 0$ the map is perfectly deterministic.

We consider successive iterations of the stochastic map, Eq. (2) with r at the critical value r_c, following techniques introduced by Feigenbaum. Setting the origin of coordinates to the x for which the function $f(x;r)$ is a maximum and rescaling this maximum to 1, the 2^nth iterate of $f(x;r_c)$ converges to $(-\alpha)^{-n} g(\alpha^n x)$, where $g(x)$ is a universal map satisfying the equation

$$g(g(x)) = -\alpha^{-1} g(\alpha x) \tag{3}$$

with $\alpha = -1/g(1)$. Adding a small amount of noise $\xi\sigma$, we assume that the corresponding 2^nth iterate of the map converges to $(-\alpha)^{-n}[g(\alpha^n x) + \xi\sigma\kappa^n D(\alpha^n x)]$ with $D(x)$ a universal x-dependent noise amplitude function and κ a constant. When σ is small enough, we have

$$\begin{aligned} g(g(x)+\xi\sigma D(x)) + \xi'\sigma D(g(x)+\xi\sigma D(x)) &\approx g(g(x)) + \xi\sigma g'(g(x))D(x) + \xi'\sigma D(g(x)) + O(\sigma^2) \\ &= g(g(x)) + \xi''\sigma\{[g'(g(x))D(x)]^2 + [D(g(x))]^2\}^{1/2}. \end{aligned} \tag{4}$$

In going to the last line we used the fact that ξ and ξ' are independent random variables, and that ξ'' is also a random variable. This and our above assumption implies that $D(x)$ must satisfy the eigenvalue equation

$$KD(\alpha x)=\alpha\{[g'(g(x))D(x)]^2+[D(g(x))]^2\}^{1/2}. \quad (5)$$

We have solved Eq. (5) for the eigenvalue κ and the corresponding eigenfunction $D(x)$ using the known results[1] for α and $g(x)$. Carrying out a calculation involving a polynomial interpolation for $D(x)$ we have found $\kappa=6.619\,03\ldots$.

In the immediate vicinity above the transition to chaos the invariant probability distribution associated with the stochastic map will consist of 2^n bands, where n is an integer that grows in the case of the deterministic map by unit steps to infinity as the transition is approached.[8,9] In the case of the stochastic map, n grows to a finite value—and then decreases by unit steps as one passes to the other side of the transition. This modification of the deterministic bifurcation sequence is called a bifurcation gap.[6]

We now extend to the present case the previous discussion in Ref. 2 of the scaling behavior of the Liapunov characteristic exponent $\bar{\lambda}$, given by

$$\bar{\lambda}=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\ln|f'(x_k;r)|, \quad (6)$$

or alternatively

$$\bar{\lambda}=\int p(x)\ln|f'(x;r)|\,dx, \quad (7)$$

where $p(x)$ is the invariant probability distribution associated with the map. Applying the abovementioned considerations we obtain[10]

$$\bar{\lambda}=2^{-n}L(\delta^n\bar{r},\ \kappa^n\sigma). \quad (8)$$

Now, we assume that there will be 2^n bands in the chaotic regime when $\kappa^n\sigma$ is of order unity so that $n=-\ln\sigma/\ln\kappa$. Substituting this result into Eq. (9) we obtain Eq. (1) for $\bar{\lambda}$ with the two exponents θ and γ given in terms of Feigenbaum's eigenvalue δ and the new eigenvalue κ by $\theta=\ln 2/\ln\kappa=0.366\,754\ldots$ and $\gamma=\ln\delta/\ln\kappa=0.815\,359\ldots$. The appearance of a bifurcation gap implies that $L(y)$ vanish at some $y=y_0$ which in turn implies that the maximum number n of bifurcations is determined by the relation $\bar{r}_{n\,\max}=y_0\sigma^\gamma$. This behavior has been observed numerically.[6]

Measurements of the behavior of $\bar{\lambda}$ as a function of σ at $\bar{r}=0$ have already been made by numerically calculating $\bar{\lambda}$ according to Eq. (6) with varying amounts of noise.[7] The measured value for θ is 0.37 ± 0.01. This agrees with our theoretical value for θ to within the experimental error.

To verify the existence of the scaling function $L(y)$ of Eq. (1) we used our values of θ and γ to plot $\bar{\lambda}\sigma^{-\theta}$, with $\bar{\lambda}$ the result of numerical calculations of Eq. (6), as a function of $\bar{r}\sigma^{-\gamma}$. The results are shown in Figs. 1 and 2 for three different noise levels: $\sigma=10^{-6}$, 10^{-8}, and 10^{-10}. The results for those three different noise levels all fall on a universal curve in the chaotic regime,

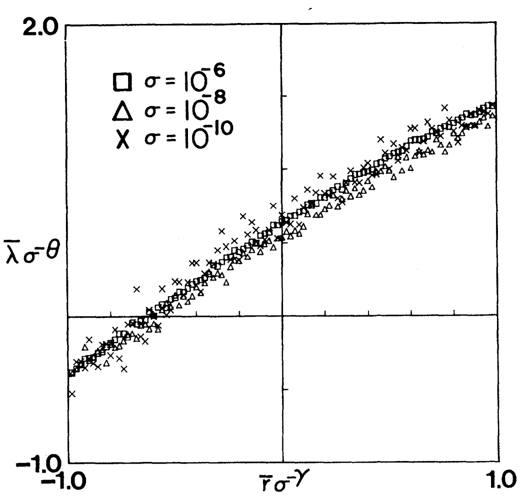

FIG. 1. Numerical determination of the scaling function $L(y)$, Eq. (1). The quantity $\bar{\lambda}\sigma^{-\theta}$ is plotted against 100 values of $y=\bar{r}\sigma^{-\gamma}$ at each of three noise levels: $\sigma=10^{-6}$, 10^{-8}, and 10^{-10}. $\bar{\lambda}$ was calculated with use of Eq. (6), with $N=10^6$ and with ξ_K a uniformly distributed random number of standard deviation σ.

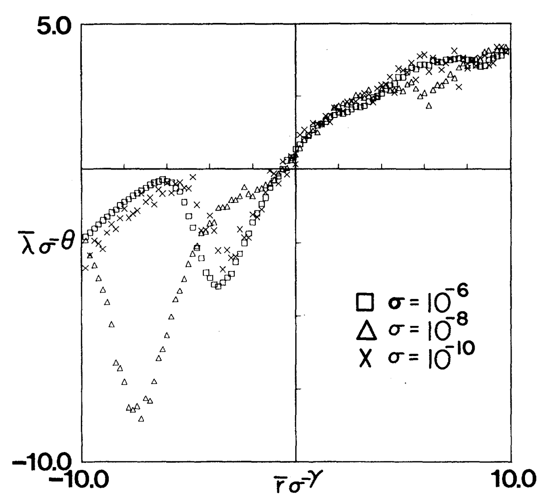

FIG. 2. $\bar{\lambda}\sigma^{-\theta}$ is plotted again, but over a wider range of $y=\bar{r}\sigma^{-\gamma}$ to illustrate the scaling regime. See text for discussion of various features. The details are the same as in Fig. 1, expect that $\bar{\lambda}$ was calculated with $N=10^5$ in Eq. (6).

and in its immediate vicinity, Fig. 1, and fit the asymptotic behavior $L(y) \sim y^{\tau}$ for large y. The results do *not* coincide in the periodic regime, Fig. 2, but they could have been made to agree if we had chosen noise amplitudes differing by factors of κ, instead of factors of 100. This more restricted scaling follows from considerations of the type enunciated above.

These results appear to us to be both exciting and highly provocative. A theoretical picture of the transition to turbulence is just beginning to emerge; the analogy to critical phenomena should lead to new and important insights into the nature and characteristics of this transition.

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[10]The details of the derivation of the result (8), which involves a careful consideration of the structure of the bands, will be presented in a future paper.

Scaling Theory for Noisy Period-Doubling Transitions to Chaos

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The effect of noise on systems which undergo period-doubling transitions to chaos is studied. With the aid of nonequilibrium field-theoretic techniques, a correlation-function expression for the Lyapunov parameter (which describes the sensitivity of the system to initial conditions) is derived and shown to satisfy a *scaling theory.* Since these transitions have previously been shown to exhibit *universal behavior*, this theory predicts *universal effects* for the noise. These predictions are in good agreement with numerical experiments.

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During the past few years, the onset of chaotic behavior, after a sequence of period-doubling transitions, has been extensively studied. Feigenbaum[1] has observed that these transition sequences exhibit "universal" features akin to those of phase transitions; Collet and Eckmann[2] have noted that these universal features are shared by differential equations and multidimensional maps in which chaos is preceded by a sequence of period doublings; and Libchaber and Maurer[3] have observed this phenomenon in a convective cell with small aspect ratio. Recently, Huberman and Rudnick[4] have related one of the pretransitional parameters identified by Feigenbaum with the growth of disorder (i.e., the Lyapunov parameter) in the chaotic regime, and Huberman and Crutchfield[5] have examined numerically the effect of external noise on the onset of chaos. Nevertheless, many connections between period-doubling chaotic transitional phenomena and the critical phenomena at second-order phase transitions remain unclear.

The purpose of this Letter is the following: (1) to present a scaling theory (in which "noise" and "stress" play the role of external field and temperature) for systems that become chaotic