experiment to realize the optical tristability in which sodium vapor is used as a dispersive medium. By filling He gas at pressure higher than 200 Torr as a buffer gas, γ_{ab} for D_1 line at 589.6 nm becomes larger than 2 GHz,[6] and we can neglect hole burning effect and hyperfine pumping especially for off-resonant light. Furthermore, the buffer gas mixes the excited hyperfine and Zeeman structure completely. Thus the situation is very close to the model which we have used in this paper. To satisfy the inequality (13), $2\kappa L$ must be of the order of unity or larger, which can be achieved by choosing $N \sim 10^{12}$ cm^{-3}, $L = 10$ cm, and $|\Delta| = 30\gamma_{ab}$. Then the absorption loss $2\alpha L$ is about 0.1 and will be neglected. The required optical power density of a cw dye laser is of the order of 10 mW/mm^2.

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Instability of the Brillouin-Flow Equilibrium in Magnetically Insulated Structures

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Presented herein is a fully electromagnetic and relativistic stability analysis of the Brillouin-flow equilibrium for magnetic insulation in planar geometry. Instability of TM waves propagating in the direction of the sheared electron flow is found. This instability occurs at short wavelengths at frequencies above the cyclotron and plasma frequencies relevant to the system. It is found that relativistic effects can make the maximum instability growth rate normalized to the cyclotron frequency substantially lower than the nonrelativistic value (0.06).

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In current electron and light-ion–driven inertial-confinement fusion schemes, transmission lines capable of carrying power densities of the order of 1 TW/cm^2 at electric field levels exceeding 5 MV/cm are required.[1] These stresses, far exceeding the standoff capabilities of conventional insulators, necessitate the use of a magnetic field applied perpendicularly to the electric field in a gap to prevent breakdown by electrons. Known as magnetic insulation, this method of breakdown inhibition also finds useful application in the production of intense ion beams in vacuum diodes, in relativistic magnetrons, and in multiple-stage linear accelerators for charge-neutralized ion beams. An examination of the linear stability of the magnetically insulated state is of interest in helping to determine, as a function of system parameters, the length of transmission line over which breakdown should be inhibited (or the time duration of the insulation), the quality of an ion beam which passes through an insulated electron layer, or the linear startup state of a magnetron device. In this paper, some results of the first solution of a fully relativistic and electromagnetic treatment of the stability of the magnetically insulated Brillouin-, or laminar-, flow state[2] are presented. As the name implies, electrons emitted from a cathode into this state are confined to a sheath near the cathode in which they drift laminarly along equipotentials at the local, self-consistent $\vec{E} \times \vec{B}$ drift velocity (which is sheared monotonically).

It is found that TM waves propagating along the direction of electron flow are unstable to perturbations at frequencies above the plasma and cyclotron frequencies in the electron sheath (which, when calculated relativistically, are equal and homogeneous across the sheath) and for all wavelengths shorter than some maximum value. The source of the instability is a resonant interaction between the wave and those electrons within the sheath having velocities such that a relativistic, Doppler-shifted plasma resonance occurs. This instability is to be distinguished from the longer-wavelength diocotron instability, which dominates in low-density beams[3] (plasma frequency, $\omega_p \ll \Omega_c$, the cyclotron frequency), but which is stabilized by the conducting cathode bounding the sheath.[4] In fact, the instability mechanism is found to be identical to that of the short-wavelength instability found by earlier workers[3,5] investigating crossed field beams at nonrelativistic velocities and in the absence of bounding conducting surfaces (although this analysis was later generalized to include relativistic effects,[6] no solutions for the resulting equations were presented).

To briefly review the equilibrium state[2,7] prior to a discussion of the stability analysis, consider a planar system infinite along the y and z axes. The electron sheath extends from the cathode ($x=0$) to x_*, while there is a vacuum region between x_* and the anode ($x=d$). Allowing quantities to vary only with x, the equilibrium is derived from the cold-fluid equations with use of the conservation laws for energy and canonical momentum as well as Poisson's equation and Ampere's law. It is assumed that the voltage is turned on slowly compared to an electron cyclotron period and that emission from the cathode is space-charge limited. The state can be characterized by two parameters: the sheath thickness, x_*, and an inverse scale length related to the equilibrium magnetic field at the cathode, $\beta = e|B(x=0)|/mc^2$, where $-e$ and m are the charge and mass of an electron, c is the speed of light, and the magnetic field $\vec{B}_0$ lies along $\hat{y}$. In the sheath, $\vec{E}_0 \equiv$(electric field)$=\hat{x}(-mc^2\beta/e)\sinh(\beta x)$, $\vec{B}_0 = \hat{y}(-mc^2\beta/e)\cosh(\beta x)$, n_0 (electron number density)$=(mc^2\beta^2/4\pi e^2)\cosh(\beta x)$, v_{z0} (electron velocity)$=c\tanh(\beta x)$, $\gamma_0=(1-v_{z0}^2/c^2)^{-1/2}=\cosh(\beta x)$, and $\omega_p^2 = 4\pi n_0 e^2/m\gamma_0 = \Omega_c^2 = (eB_0/m\gamma_0 c)^2 = \beta^2 c^2$. In the vacuum, E_0 and B_0 are fixed at the sheath values for $x=x_*$. Integrating E_0 and B_0 across the gap, one can calculate the voltage, φ_a, and magnetic flux, A_a, at the anode, which must obey the insulation condition[7] (i.e., $x_* \le d$) $e\varphi_a/mc^2 + 1 \le (eA_a/mc^2)^2 + 1$.

The instability is caused by a TM wave along $\hat{z}$, the direction of electron flow. All quantities of interest are expressed as an equilibrium quantity plus a small amplitude perturbation; e.g., $\vec{E} = \hat{x}E_0 + \delta\vec{E}(x)\exp[i(kz-\omega t)]$. The set of relevant terms is $\delta\vec{E} = \hat{x}\delta E_x + \hat{z}\delta E_z$, $\delta\vec{B} = \hat{y}\delta B_y$, $\delta\vec{v} = \hat{x}\delta v_x + \hat{z}\delta v_z$, δn, and $\delta\gamma = \gamma_0^3 v_{z0}\delta v_z/c^2$. Independent variables are normalized by defining $u = \Omega_c x/c$, $K = kc/\Omega_c$, $w_\varphi = \omega/kc$, and $q \equiv \omega_p^2/\Omega_c^2 = 1$. In the sheath region, the governing equations come from the linearized cold fluid momentum conservation equation and Maxwell's equations with $\partial/\partial t \to -i\omega$, $\partial/\partial y \to 0$, $\partial/\partial z \to ik$, and $d/dt \to \delta v_x d/dx + i(kv_{z0} - \omega)$. The resulting set of six equations in six unknowns can be reduced to a single second-order equation for the quantity

$$F \equiv [K\gamma_0(w_{z0} - w_\varphi)]^{-1}\delta E_z$$

(where $w_{z0} = v_{z0}/c$) which obeys

$$[\Delta(u)F']' - K^2(1-w_\varphi^2)\Delta(u)F = 0, \qquad (1)$$

with $\Delta = q - K^2\gamma_0^2(w_{z0}-w_\varphi)^2$ and d/du denoted by a prime. Because $\delta E_z(u=0)=0$, $F(0)=0$. In the vacuum region, Maxwell's equations and the constraint that $\delta E_z = 0$ at the anode ($u=u_a$), imply that

$$\delta E_x = iA_v(1-w_\varphi^2)^{-1/2}\cosh[K(1-w_\varphi^2)^{1/2}(u_a-u)],$$
$$\delta E_z = i\,\delta E_x'/K, \quad \text{and} \quad \delta B_y = w_\varphi \delta E_x,$$

with A_v a constant.

To complete the formal solution to the problem, it remains to match sheath and vacuum solutions at the boundary between the two regions. Because $\delta\vec{v}$ has a component normal to the equilibrium boundary at u_*, the perturbation will sinusoidally ripple the boundary with an amplitude $\xi = -i\{K\times[w_{z0}(u_*) - w_\varphi]\}^{-1}\delta v_x(u_*)/c$. Linearizing and expanding the equilibrium fields about their values at u_*, the two independent matching conditions are

$$[\delta E_x(u_*)]_{\text{vac}} = [\delta E_x(u_*) + \xi(dE_0/du)(u_*)]_{\text{sheath}},$$

and

$$[\delta E_z(u_*)]_{\text{vac}} = [\delta E_z(u_*)]_{\text{sheath}},$$

where the sheath fields and ξ are derivable in terms of F in Eq. (1). Taking a ratio of the two terms and manipulating somewhat yields a dispersion relation:

$$D(K, w_\varphi) \equiv (1 - w_\varphi^2)^{1/2} \tanh[K(1 - w_\varphi^2)^{1/2}(u_a - u_*)] [2\Delta(u_*)F^{-1}(u_*)F'(u_*) + \Delta'(u_*)] - 2K\gamma_*^2[1 + k^2(1 - w_\varphi^2)](w_{z*} - w_\varphi)^2 = 0, \quad (2)$$

where $\gamma_* = \cosh u_*$, $w_{z*} = \tanh u_*$, and $F^{-1}(u_*)F'(u_*)$ is determined from (1). The solution of Eq. (2) is possible analytically only in the limit of very small K under the assumption that $Kw_\varphi \to 0$ as $K \to 0$. In this case, one finds[8] two real solutions (indicating stable, propagating oscillations), $w_\varphi^+ > w_\varphi^-$, which are slow waves ($|w_\varphi| < 1$). The solution for arbitrary values of K is more complicated since (1) is not solvable analytically in general, and is, in fact, singular for certain ranges of w_φ which make $\Delta(u) = 0$ for $0 \le u \le u_*$. These singularities for $\Delta = 0$ occur when $\omega = kv_{z0} \pm \omega_p/\gamma_0$ (i.e., at layers where a relativistic Doppler-shifted plasma resonance occurs) and lead to a continuous spectrum of localized oscillations which decay secularly (as powers of t) at long times.[8]

Equation (2) describing the system's discrete collective oscillations has been solved numerically with use of Muller's method to find w_φ for $K < 1$ and a contour integration routine based on Cauchy's theorem[8] for $K \ge 1$. One finds that from a stability standpoint, the modes of most interest are those which correspond to the analytically derivable modes as $K \to 0$, in particular the mode denoted w_φ^-. With $w_\varphi = w_r + iw_i$ and $\Omega = Kw_\varphi = \Omega_r + i\Omega_i$, plots of w_r vs K are shown in Fig. 1 for three representative equilibrium values of u_a and u_*. The stable mode, w_φ^+, decreases with K until at some point this root of Eq. (2) migrates through the continuous spectrum branch cut and transfers its contribution to system response to the continuous spectrum. The unstable mode, w_φ^-, is approximately the $K = 0$ value of w_φ^- for small k and approximately $w_{z0}(u_*)$ for large K. Calculation of w_i^- reveals that the onset of instability occurs within the transition region where $K \sim 1$, and that the system is unstable for all equilibrium states defined by u_a and u_* and for all K greater than the value of K at the onset of instability. The general behavior is illustrated by the particular case of $u_a = 4.0$, $x_*/d = 0.5$ plotted in Fig. 2: a rapid initial increase of w_i^- with K, followed by a decrease roughly as K^{-1} so that Ω_i^- remains almost constant. This broadband nature can be understood by bearing in mind the fact that the mechanism which transfers energy from the drifting electrons to the wave is an interaction between a collective oscillation, w_φ^-, and a localized disturbance at a resonance layer where $\omega_r^- = kv_{z0} + \omega_p/\gamma_0$. At long wavelengths, $w_r^- \lesssim K^{-1}$ and $\omega_r^- < \omega_p$, and so there is no resonance layer in the sheath and no instability; at short wavelengths such that $w_r^- \gtrsim K^{-1}$ and $\omega_r^- \ge \omega_p$, however, the resonance layer always lies within the sheath.

At nonrelativistic parameter values ($u_* \ll 1$), the maximum Ω_i is 0.06, corresponding to a maximum of w_i^- at $ku_* = 2.05$, and $w_i^-/w_r^- = 0.033$, in keeping with the earlier electrostatic work on crossed field beams.[3] Advances in high-voltage technology, however, have shifted interest to the

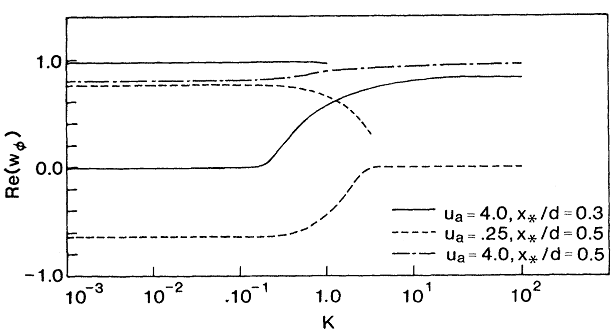

FIG. 1. Real values of $w_\varphi = \omega/kc$ vs $K = kc/\Omega_c$ for slow TM waves in three representative equilibrium states. The curve of Re(w_φ) for u_a = 4.0, x_*/d = 0.5 corresponding to ω_φ^+ as $K \to 0$ is not shown separately, since it effectively coincides with the uppermost curve (solid line for u_a = 4.0, x_*/d = 0.3).

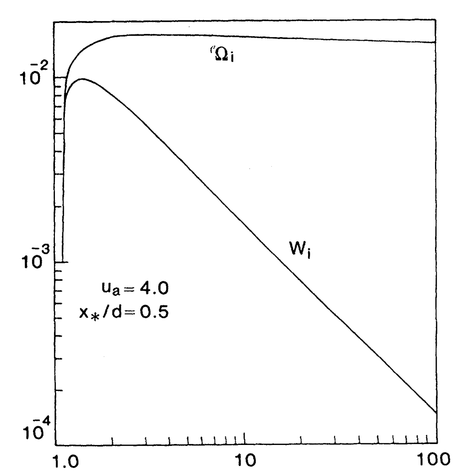

FIG. 2. Imaginary values of w_φ and $\Omega = \omega/\omega_p$ vs K for $u_a = \beta d$ = 4.0 and x_*/d = 0.5.

relativistic regime, where it is necessary to know the scaling of growth rates as u_* and u_a (or φ_a and A_a) increase. Although a complete presentation of all results must be deferred to a longer paper,[8] several broad conclusions can be formulated on the basis of calculated solutions to Eq. (2) over a wide range of equilibrium parameters. First, we find the important result that relativistic effects can substantially decrease Ω_i^- (growth rate normalized to ω_p or Ω_c) below 0.06, a result which is encouraging from a pulsed power, inertial confinement fusion standpoint. For example, $\Omega_i \cong 0.01$ at $u_a = 5$, $x_*/d = 0.5$.[8] Secondly, for fixed gap spacings, ω_i is seen to scale roughly with the velocity shear at the sheath edge, $dv_{z0}/dx = \omega_p \operatorname{sech}^2 \beta x_*$. This leads to several conclusions: (1) at small $u_a = d$, ω_i increases linearly through ω_p, (2) at large u_a, ω_i decreases rapidly, and (3) systems having $\varphi_a \ll A_a$ are among the most unstable, since this condition can be shown to increase the shear. This last conclusion correlates well with the experimental observation[9] on the MITE (Magnetically insulated Transmission Experiment) generator at Sandia National Laboratories that magnetically insulated lines carrying power to short circuits (which implies $\varphi_a \ll A_a$) experience much greater losses than for lines carrying power to loads of greater impedance. Third, defining an instability growth length by dividing the growth rate into the group velocity, one finds this length to vary monotonically from a fraction of the gap for thin sheaths ($x_*/d \ll 1$) to many times d as $x_*/d \to 1$. The implication is that thick sheaths are more stable than thin ones, and the reason is that the conducting anode provides a progressively stronger stabilizing influence as the sheath edge approaches it (similar to the effect by which the cathode stabilizes the diocotron mode). Finally, similar calculations[8] for two additional classes of waves, TE waves along $\tilde{z}$ and waves parallel to B_0, are found to be stable, so that waves propagating at some angle to $\tilde{z}$ should have reduced growth rates, going to zero for $\vec{k}\cdot\hat{z} = 0$.

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