# Radiation of Energy and Change in Color of a Point Source of the Yang-Mills Field

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It is shown that the color of a single, classical pole particle can change only if the gauge group contains two generators $a$ and $b$ such that $[a,b]=a\neq 0$ and the particle is unaccelerated. If the gauge group is compact and semisimple, then color is constant, but there is transfer of energy by the Yang-Mills field, in complete analogy with the situation in electrodynamics described by the Liénard-Wiechert potentials.

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Classical solutions to Yang-Mills equations are probably not as physically relevant as the corresponding solutions in Maxwell's theory, but their study may elucidate some general features of strong and weak interactions.[1] It is interesting, in particular, to consider the similarities and differences among analogous situations in electrodynamics, in non-Abelian gauge theories of the Yang-Mills type, and in Einstein's theory. The latter may be looked upon as a gauge theory of the gravitational field, but with different equations and a structure richer than in other gauge theories.[2]

Color in chromodynamics plays a role analogous to that of the charge in electrodynamics and of energy-momentum in Einstein's theory. One can also compare color and charge to spin in the Einstein-Cartan theory,[3] or in other theories of gravity with torsion,[4] but this analogy is not made use of here. In a sense, color is more like energy-momentum in Einstein's theory than charge in Maxwell's: Gluons carry color and gravitons have energy-momentum, whereas photons are neutral. Therefore, one can formulate the question of radiation of color, analogous to the problem of gravitational radiation: Is it possible for a bounded classical source of the gluon field to change its total color? Of course, one should be careful about distinguishing "real" changes of color from those induced by the freedom in the choice of gauge.

This Letter answers the question of color radiation—essentially in the negative—for the rather special case of a point source with the structure of a simple pole. Its field is assumed to be derivable from the Liénard-Wiechert potential adapted by Arodź to the Yang-Mills theory.[5] It is shown that—against the expectations of Drechsler[6]—such a point particle does radiate energy when accelerated. Its color can change only if the particle is unaccelerated and the gauge group $G$ admits two generators $a$ and $b$ such that $[a,b]=a\neq 0$.

It is convenient to represent the Yang-Mills potentials $(A_\mu{}^i)$ and field strengths $(F_{\mu\nu}{}^i)$ by Lie-algebra–valued differential forms[7]

$$A=A_\mu{}^i dx^\mu e_i,\quad F=\tfrac{1}{2}F_{\mu\nu}{}^i dx^\mu\wedge dx^\nu\, e_i,$$

where $(e_i)$ is a linear basis of the Lie algebra of the gauge group and Greek indices, ranging from 0 to 3, refer to space-time. The relation between $F$ and $A$ is

$$F=dA+\tfrac{1}{2}[A,A],$$

where $d$ is the exterior derivative and the bracket denotes both the Lie-algebra product and the wedge (skew) product of forms. Introducing the Hodge (dual) operator $*$ one can write Yang-Mills equations as

$$D*F:=d*F+[A,*F]=4\pi *j,$$

where $*j$ is the three-form of color current of the source, dual to $j=e_i j_\mu{}^i dx^\mu$. Similarly as in Einstein's theory, the source current satisfies $D*j=0$, but is not truly conserved by itself. The total current

$$*J:=*j-(1/4\pi)[A,*F]=(1/4\pi)d*F$$

is conserved, $d*J=0$, but is locally ill-defined because the field contribution to $*J$ is highly gauge dependent. For a bounded source, however, one can presumably find a gauge such that $A$ and $F=O(r^{-1})$ for $r\to\infty$ and total color may then be defined by Gauss's law

$$\int_{B_R}*J=(1/4\pi)\int_{S_R}*F\quad(R\to\infty),$$

where $S_R$ is the surface of the ball $B_R$ of radius $R$. The integral converges for $R\to\infty$ provided the "electric part" of the $r^{-1}$ term in $F$ is tangent to $S_R$.

Let $z^\mu(s)$ be the Cartesian coordinates of a smooth, timelike world line $z$ in a flat space-time with Minkowski metric $(g_{\mu\nu})$. The world line is parametrized by its proper time, $g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu=1$,

 

$\dot{z}^\mu = dz^\mu/ds$, $\dot{z}^0 > 0$. One associates with $z$ a system of comoving spherical coordinates $(t, r, \theta, \varphi)$ by writing[8]

$$x^\mu = z^\mu(u) + r l^\mu(\theta, \varphi)/p(u, \theta, \varphi),$$

where $u = t - r$ is a retarded time,

$$r = g_{\mu\nu}[x^\mu - z^\mu(u)]\dot{z}^\nu(u) \geq 0$$

is a radial distance measured in the rest frame of an observer moving along $z$, and $l = (l^\mu)$ is the null vector field,

$$l = (1,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta).$$

These definitions imply

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = (1 - rp^{-1}\dot{p})\, du^2 + 2\, du\, dr - p^{-2}r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

where a dot denotes differentiation with respect to $u$. The function $p = g_{\mu\nu}\dot{z}^\mu l^\nu$ has a simple physical interpretation: if $\omega$ is the frequency of a beam of light moving in the $(\theta, \varphi)$ direction, as seen from rest in the coordinate system $(x^\mu)$, then $\omega p(u, \theta, \varphi)$ is the frequency of the same beam measured at $z^\mu(u)$ by the observer moving along $z$. The world line $z$ is straight, $\ddot{z}^\mu = 0$, if and only if $\dot{p} = 0$.

Consider a point particle of (dimensionless) color $q(s) = q^i(s)e_i$, moving along $z$. Its Liénard-Wiechert potential is defined by

$$A = q(u)r^{-1}\dot{z}_\mu(u)\, dx^\mu. \tag{1}$$

This form of the potential fixes the gauge almost completely; the only remaining freedom is of constant gauge transformations, $q \to g^{-1}qg$, $g \in G$. Such global gauge transformations are not enough to align a time-dependent color along a fixed direction, contrary to what is asserted in Ref. 5.

The field strengths corresponding to (1),

$$F = qr^{-2}\, du \wedge dr + r^{-1}\, du \wedge [\dot{q}\, dr + qr\, d(p^{-1}\dot{p})], \tag{2}$$

satisfy the Yang-Mills equation $D * F = 0$ $(r \neq 0)$ if and only if

$$\ddot{q} + [q, \dot{q}] = 0 \tag{3}$$

and

$$\partial(p^{-1}\dot{q})/\partial u = 0. \tag{4}$$

If $G$ is either (i) Abelian, or (ii) compact and semisimple, then Eq. (3) implies $\dot{q} = 0$. It is worth noting that, in the important case (ii), strict conservation of color follows from the Yang-Mills equation alone. Moreover, the field (2) has the same structure as in electrodynamics: It contains a Coulomb-like $r^{-2}$ term and a radiative $r^{-1}$ term, linear in $(\ddot{z}^\mu)$. Clearly, the latter term gives rise to outgoing radiation of energy. The expressions for the Poynting vector and the total intensity may be obtained from the corresponding formulas derived in electromagnetism by replacing the square of the electric charge by $-\mathrm{Tr}q^2$. Moreover, it is easy to see that, if $\dot{q} = 0$, then the color current $*j$ corresponding to (1) is a distribution with support on the world line $z$. It can be expressed in a form used in the context of the Yang-Mills theory by Wong[9] and by Adler,[10]

$$j_\mu{}^i(x) = q^i \int \dot{z}_\mu(s)\delta(x - z(s))\, ds,$$

where $\delta$ is the four-dimensional Dirac function.

If $\dot{q} \neq 0$, then Eq. (4) leads to

$$\dot{p} = 0 \quad \text{and} \quad \ddot{q} = 0.$$

The particle is thus unaccelerated and its color changes linearly with time,

$$q(t) = at + b, \tag{5}$$

where, by virtue of Eq. (3),

$$[a, b] = a \neq 0.$$

The $x$ coordinates may now be adjusted so that $p = 1$ and the solution assumes a manifestly spherically symmetric form,

$$A = r^{-1}(au + b)\, dt, \tag{6}$$

$$F = r^{-2}(at + b)\, dt \wedge dr. \tag{7}$$

In this case color changes, but there is no transfer of energy by the gluon field. Incidentally, the field strengths (7) may also be derived from the potential $r^{-1}(at + b)\, dt$ which is not gauge equivalent to (6).

Gauss's law applied to the field strengths (7) gives the time-dependent charge (5). A closer analysis shows, however, that the gauge configuration described by (6) and (7) is, in fact, time-independent. This is easily seen in the simple, but typical, case when $G = \mathrm{SL}(2, R)$, and

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $S = I + au$, then $S^{-1} = I - au$ and

$$\begin{aligned} A' &= S^{-1}AS + S^{-1}dS = r^{-1}b\, dt + a\, du, \\ F' &= S^{-1}FS = r^{-2}(ar + b)\, dt \wedge dr. \end{aligned} \tag{8}$$

The gauge-transformed field (8) is explicitly time independent, but contains an unexpected $r^{-1}$ term which makes it impossible to apply Gauss's law.

Returning to the analogy between the Yang-Mills and Einstein theories, one should bear in mind that a single, spherically symmetric body cannot radiate gravitational waves, but a system of bodies, moving under their mutual attraction, is believed to lose energy because of gravitational radiation. Is it true that, similarly, a system of particles can radiate color, whereas a single particle cannot? An approximate computation, which will be reported in a forthcoming paper, confirms this conjecture. It appears that radiation of color by a system of classical point sources is associated with the noncommutativity of their color charges, a phenomenon whose significance was emphasized by Adler.[10]

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# Precise Measurement of the $\Xi^0$ Magnetic Moment

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The magnetic moment of the $\Xi^0$ hyperon has been measured to be $\mu_{\Xi^0} = -1.253 \pm 0.014$ nuclear magnetons. A new measurement of $\mu_\Lambda$ is also reported.

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The magnetic moments of the members of the baryon octet were first calculated under the assumption of perfect SU(3) symmetry by Coleman and Glashow,[1] who obtained values for the static moments, $\mu_\Lambda$, $\mu_{\Sigma^+}$, $\mu_{\Sigma^0}$, $\mu_{\Sigma^-}$, $\mu_{\Xi^0}$, $\mu_{\Xi^-}$, and the transition moment, $\mu_{\Sigma\Lambda}$, in terms of the magnetic