a function of p_Σ, the momentum of the Σ^+, are plotted in Fig. 4, along with comparable data for Λ.[6] The magnitude of the Σ^+ polarization is the same as that of the Λ at the same momentum. The most remarkable feature of this new data is that the sign of the polarization is opposite that of Λ.

The simple quark model referred to above assumes that one quark in the incident proton is lost through a hard collision, leaving a spectator diquark (*uu* or *ud*) which then picks up an *s* quark to form the forward out-going hyperon. Thus $uud \rightarrow uus$ produces a Σ^+, while $uud \rightarrow uds$ produces either a Λ or a Σ^0. Assume that the *s* quark is polarized by some unspecified mechanism. Then $P_\Lambda = P_s$, because the (*ud*) spectator is in a singlet state. For the Σ^+ and Σ^0 the nonstrange quarks must be in a triplet state, so that the polarization of the composite baryon is opposite to that of the strange quark: $P_{\Sigma^+} = P_{\Sigma^0} = -\frac{1}{3}P_\Lambda$. The observed sign reversal is thus expected from the model, although the predicted magnitude $|P_{\Sigma^+}| = \frac{1}{3}|P_\Lambda|$ is smaller than the measured one.

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Construction of Exact Yang-Mills-Higgs Multimonopoles of Arbitrary Charge

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Exact axially symmetric multimonopole solutions of arbitrary topological charge are presented and their relevant features discussed.

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Ever since the pioneering work of 't Hooft and Polyakov[1] there has been considerable interest in static finite-energy magnetic-monopole solutions of classical Yang-Mills-Higgs gauge theories. More than five years ago, in the limit of vanishing Higgs potential, an exact analytic charge-one monopole solution was discovered.[2] Since then many people have tried, unsuccessfully, to find exact analytic multimonopole solutions. However, recently Ward[3] made a major breakthrough by presenting for the first time an exact analytic charge-two monopole solution. Inspired by Ward's work, we have now constructed, for the first time, exact monopole solutions of arbitrary charge. It is this construction we wish to describe in this Letter. The calculations leading to our construction will be briefly sketched. The details are quite involved and are presented elsewhere.[4]

Let us define in four-dimensional Euclidean space (x_1, x_2, x_3, x_4) the SU(2) gauge potentials $A_\mu{}^a$ when $a = 1, 2, 3$ and $\mu = 1, 2, 3, 4$. The gauge field strength is defined by

$$F_{\mu\nu}{}^a \equiv \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + \epsilon^{abc} A_\mu{}^b A_\nu{}^c. \qquad (1)$$

Multimonopole solutions, with magnetic charge $n = 1, 2, 3, \ldots$, may be found[5,6] within the framework described in Refs. 5 and 6. This means we want to solve the self-duality equations:

$$F_{\mu\nu}{}^a = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}{}^a \qquad (2)$$

(our convention is $\epsilon_{1234}\equiv+1$) with the requirement that $A_\mu{}^a$ be static (independent of x_4), real, and regular. We also require that $A_4{}^aA_4{}^a\to 1-2n/r+O(r^{-2})$ as $r\equiv(x_1{}^2+x_2{}^2+x_3{}^2)^{1/2}\to\infty$. (Note that $A_4{}^a$ is just the Higgs field.) Provided these conditions are met the solutions will correspond to finite-energy, $E=\frac{1}{4}\int F_{\mu\nu}{}^aF_{\mu\nu}{}^a d^3x=4\pi n$, magnetic-monopole solutions with magnetic charge n.

Yang[7] has shown that by introducing complex coordinates

$$\sqrt{2}p=x_1+ix_2,\quad \sqrt{2}\bar{p}=x_1-ix_2,$$
$$\sqrt{2}q=x_3-ix_4,\quad \sqrt{2}\bar{q}=x_3+ix_4, \tag{3}$$

and by choosing a certain gauge, the R gauge, any solution of Eq. (2) can be brought to the following form $[A_\mu\equiv(\sigma^a/2i)A_\mu{}^a]$:

$$A_u=\begin{pmatrix}-\frac{\varphi_u}{2\varphi} & 0\\ \frac{\rho_u}{\varphi} & \frac{\varphi_u}{2\varphi}\end{pmatrix},\quad A_{\bar{u}}=\begin{pmatrix}\frac{\varphi_{\bar{u}}}{2\varphi} & -\frac{\bar{\rho}_{\bar{u}}}{\varphi}\\ 0 & -\frac{\varphi_{\bar{u}}}{2\varphi}\end{pmatrix}, \tag{4}$$

where $u=p,q$ and φ, ρ, $\bar{\rho}$ satisfy the following coupled equations:

$$(\partial_p\partial_{\bar{p}}+\partial_q\partial_{\bar{q}})\ln\varphi+\varphi^{-2}(\rho_p\bar{\rho}_{\bar{p}}+\rho_q\bar{\rho}_{\bar{q}})=0, \tag{5a}$$

$$(\varphi^{-2}\rho_p)_{\bar{p}}+(\varphi^{-2}\rho_q)_{\bar{q}}=0,$$
$$(\varphi^{-2}\bar{\rho}_{\bar{p}})_p+(\varphi^{-2}\bar{\rho}_{\bar{q}})_q=0. \tag{5b}$$

On the other hand, Ward,[8] using techniques of algebraic geometry and twistor theory, showed that all information of self-dual gauge fields can be "coded" into the structure of complex analytic vector bundles that are specified by a transition matrix G. In general there is no known procedure for explicitly extracting A_μ from G. However, Atiyah and Ward[9] argued that if the transition matrix G is of the following form:

$$\tilde{G}^{(n)}(\omega_1,\omega_2,\zeta)=\begin{pmatrix}\zeta^n & \tilde{\Omega}^{(n)}(\omega_1,\omega_2)\\ 0 & \zeta^{-n}\end{pmatrix}, \tag{6}$$

where $\sqrt{2}\,\omega_1=(\bar{q}-p\zeta)$, $\sqrt{2}\,\omega_2=-(q+\bar{p}\zeta^{-1})$ and ζ is a complex parameter, then one can systematically find A_μ. Corrigan *et al.*,[10] working in Yang's R gauge, started from Eq. (6) and found the following solutions of Eq. (5) for any $n=1,2,3,\ldots,$

$${}_n\tilde{\varphi}=\frac{\tilde{H}_{k-l}{}^{n\times n}}{\tilde{H}_{k-l}{}^{n-1\times n-1}},\quad {}_n\tilde{\rho}=(-1)^n\frac{\tilde{H}_{k-l-1}{}^{n\times n}}{\tilde{H}_{k-l}{}^{n-1\times n-1}},$$
$${}_n\tilde{\bar{\rho}}=(-1)^{n+1}\frac{\tilde{H}_{k-l+1}{}^{n\times n}}{\tilde{H}_{k-l}{}^{n-1\times n-1}}, \tag{7a}$$

where

$$\tilde{H}_{k-l+m}{}^{j\times j}=\begin{vmatrix}\tilde{\Delta}_m & \cdots & \tilde{\Delta}_{m-j+1}\\ \vdots & \ddots & \\ \tilde{\Delta}_{m+j-1} & & \tilde{\Delta}_m\end{vmatrix}, \tag{7b}$$

$$\tilde{\Delta}_l=(1/2\pi i)\oint\tilde{\Omega}^{(n)}(\omega_1,\omega_2)\zeta^l(d\zeta/\zeta), \tag{7c}$$

so that

$$\partial_p\tilde{\Delta}_l=-\partial_{\bar{q}}\tilde{\Delta}_{l+1}\,\partial_q\tilde{\Delta}_l=\partial_{\bar{p}}\tilde{\Delta}_{l+1}. \tag{7d}$$

It remains to determine which of the above solutions indeed give static, real, and regular gauge fields.

Ward[3] has given precise and sufficient conditions which the transition matrix $\tilde{G}^{(n)}$ must satisfy in order for the gauge potentials $A_\mu{}^a$ to be static and in some gauge real. We have constructed a class of transition matrices which fulfill Ward's requirements that they be static and real. In our construction

$$\tilde{\Omega}^{(n)}(\omega_1,\omega_2)=\exp(\omega_1+\omega_2)\left[\frac{e^{\omega}+(-1)^ne^{-\omega}}{2P_n(\omega)}\right], \tag{8}$$

where $\omega=\omega_1-\omega_2$ and $P_n(\omega)$ is a polynomial of the nth order in ω with real coefficients: $P_n{}^*(\omega)=P_n(\omega^*)$.

It remains to verify that the gauge fields are regular. For this purpose we need the following formula for the square of the Higgs field[5]:

$$h^2=A_4{}^aA_4{}^a=1-\nabla^2\ln\tilde{H}_{k-l}{}^{n\times n}. \tag{9}$$

Since h^2 is gauge invariant a necessary condition for the gauge field to be regular is that $\tilde{H}_{k-l}{}^{n\times n}\neq0$ for all (x_1,x_2,x_3,x_4). Now the class of solutions generated from the transition matrix defined by Eq. (8) can be shown to be axially symmetric and moreover mirror symmetric[11] which implies that $\tilde{H}_{k-l}{}^{n\times n}$ can be identified as the "superpotential" in terms of which all gauge invariant quantities can be expressed. Thus in order to have regular gauge fields it is both necessary and sufficient that $\tilde{H}_{k-l}{}^{n\times n}\neq0$.

We have found that to insure $\tilde{H}_{k-l}{}^{n\times n}\neq0$ on and around the x_3 axis and the $x_3=0$ plane we must assume that $\tilde{\Omega}^{(n)}$ is generated by the following "splitting rule":

$$\tilde{\Omega}^{(n)}(x_1, x_2, x_3, x_4, \zeta) = \tilde{\Omega}^{(n-1)}(x_1, x_2, x_3 - \tfrac{1}{2}i\pi, \zeta) + \tilde{\Omega}^{(n-1)}(x_1, x_2, x_3 + \tfrac{1}{2}i\pi, \zeta), \tag{10}$$

so that starting from the one-monopole solution, as given by Ward[3],

$$\tilde{\Omega}^{(1)}(\omega_1, \omega_2) = \exp(\omega_1 + \omega_2)(\sinh\omega/\omega), \tag{11}$$

we are led to

$$P_n(\omega) = [(n-1)!\ \pi^{n-1}]^{-1} \prod_{k=1}^{n} (\omega - Z_k), \tag{12}$$

where $Z_k = i\pi[(\frac{1}{2}n+1) - k]$. We then find, using Eqs. (7c) and (12)

$$\tilde{\Delta}_l = \exp(ix_4 - il\theta)(-1)^l \tfrac{1}{2}\int_{-1}^{1} dt\, \exp(-tx_3)(2\cos\tfrac{1}{2}\pi t)^{n-1}\left(\frac{1+t}{1-t}\right)^{l/2} I_l[s(1-t^2)^{1/2}], \tag{13}$$

where $s^2 = x_1^2 + x_2^2$, $\tan\theta = x_2/x_1$ and I_l is the modified Bessel function of the first kind of order l.

The integral in Eq. (13) can now be evaluated by explicitly expanding I_l in its power series and it requires some algebra to show that for $0 \leq l < n$

$$\tilde{\Delta}_{\pm l} = (-1)^l \exp(ix_4)(x_1 \pm ix_2)^{-l}(1 \mp \partial_3)^l \tilde{\Lambda}_l, \tag{14a}$$

$$\tilde{\Lambda}_l = (\tfrac{1}{2}\pi)^{1/2} \sum_{k=1}^{n} \alpha_k r_k^{l-1/2} I_{1/2-l}(r_k), \quad \alpha_k = [(n-1)!/(k-1)!(n-k)!], \tag{14b}$$

and $r_k^2 = x_1^2 + x_2^2 + (x_3 - z_k)^2$. Using the generating formula for modified spherical Bessel functions $I_{1/2-l}$ one can prove by induction the following explicit formulas in terms of elementary functions:

$$\tilde{\Delta}_l = \Delta_l, \ |l| < n; \quad \tilde{\Delta}_{\pm n} = \Delta_{\pm n} + (x_1 \pm ix_2)^{-n}[(2\pi)^{n-1}(n-1)!]\exp(ix_4 \mp x_3) \tag{15a}$$

$$\Delta_l = \exp(ix_4) \sum_{k=1}^{n} \alpha_k \left\{ \left(\frac{x_3 - z_k - r_k}{\sqrt{2}\rho}\right)^l \frac{\exp(r_k)}{2r_k} - \left(\frac{x_3 - z_k + r_k}{\sqrt{2}\rho}\right)^l \frac{\exp(-r_k)}{2r_k} \right\}. \tag{15b}$$

We can use Eq. (15) to compute $\tilde{H}_{k-l}^{n\times n}$ in the region when the exponentially damped corrections $O[\exp(-2r_k)]$ can be dropped. To do this we replace $\tilde{\Delta}_l$ by $\tilde{\Delta}_l^A$ where

$$\tilde{\Delta}_l^A = \exp(ix_4) \sum_{k=1}^{n} \alpha_k \left(\frac{x_3 - z_k - r_k}{\sqrt{2}\rho}\right)^l \frac{\exp(r_k)}{2r_k} \quad (-n < l < n) \tag{16}$$

in which case the determinants $\tilde{H}_{k-l}^{n\times n}$ (with $\tilde{\Delta}_l \to \tilde{\Delta}_l^A$) greatly simplify and can be reduced to the evaluation of Vandemonde determinants and we find

$$\tilde{H}_{k-l}^{n\times n}(\tilde{\Delta}_l \to \tilde{\Delta}_l^A) = \exp(inx_4) \prod_{j=1}^{n} \left[\alpha_j \frac{\exp(r_j)}{2r_j}\right] \prod_{i<j} \left[1 - \frac{x_3 - z_i - r_i}{x_3 - z_j - r_j}\right]\left[1 - \frac{x_3 - z_j - r_j}{x_3 - z_i - r_i}\right]. \tag{17}$$

From Eqs. (17) and (9) it follows that

$$h = 1 - \sum_{k=1}^{n} \frac{1}{r_k} + O[\exp(-2r_k)]. \tag{18}$$

In Refs. 5, 6, and 11 it is shown how to explicitly find a complex gauge transformation that turns all the fields into real fields. The resulting solutions correspond to n magnetic charges superimposed at the origin and turn out to be axisymmetric and mirror symmetric.[11] Solutions to the Ernst equations of general relativity are also generated in a straightforward way.[11] As a mathematical by-product of our results, we can also prove the formal equivalence of the $\tilde{H}_{k-l}^{n\times n}$ determinant to the partition function of a special class of U(n) invariant lattice QCD_2 theories with fermions.

Finally, we would like to mention that because of Eq. (9), h in general will not be a rational function. On the x_3 axis and on the $x_3 = 0$ plane, however, h does become rational and it turns out to be

$$h(x_1 = x_2 = 0) = \left|(\tanh x_3)^{(-1)^n} - \sum_{k=1}^{n} \frac{1}{x_3 - z_k}\right|, \tag{19a}$$

$$h(x_3 = 0) = \left|\frac{(1-\partial_3)_n \rho}{n\varphi}\right|_{x_3=0} = \left|\frac{(1+\partial_3)_n \bar{\rho}}{n\varphi}\right|_{x_3=0}. \tag{19b}$$

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