Ising Field Theory: Quadratic Difference Equations for the n-Point Green's Functions on the Lattice

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For the two-dimensional Ising model at arbitrary temperature, a system of quadratic difference equations is presented, involving the correlations of n order variables and that of $n-2$ order and two disorder variables. With suitable boundary conditions these equations specify the correlations.

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After the famous calculation of the thermodynamic properties of the two-dimensional Ising model,[1] relations between the various correlation functions were studied in the early sixties.[2] These relations have two characteristics: (a) They are linear and (b) correlation functions for n spins are coupled to those with more than n spins.

Recently another type of relations is found with different characteristics: (a) They are quadratic[3,4] and (b) they contain only the two-spin correlation function. Considered as partial difference equations these relations[4] determine the two-spin correlation function uniquely when suitable boundary conditions are adjoined. It is the purpose of the present paper to extend these quadratic recursion relations from two-spin correlations to n-spin correlations.[5] We emphasize that the difference equations connect only correlation functions of the same n, not different n's.

Historically such a nonlinear equation[6,7] was first found for the two-spin correlation function in the scaling limit, i.e., the limit where $T\to T_c\pm$ and the distances between spins are $O(|T-T_c|^{-1})$. More specifically, in this limit the underlying equation is the Painlevê differential equation of the third kind.[8] The generalization to the n-spin correlation function in the scaling limit was carried out in a remarkable series of papers by Sato, Miwa, and Jimbo.[9] Therefore, two special cases of the difference equations to be presented here are previously known: (a) $n=2$ for arbitrary T and (b) the scaling limit for arbitrary n.

The interaction energy of the Ising model on a square lattice with completely arbitrary interaction energies is

$$\mathcal{E}=\sum_{j,k}[E_1(j,k)\sigma_{j,k}\sigma_{j,k+1}+E_2(j,k)\sigma_{j,k}\sigma_{j+1,k}], \tag{1}$$

where $\sigma_{j,k}=\pm 1$. We will write equations involving the correlation of n spins $\sigma_{M,N}$ and the correlation of $n-2$ spins $\sigma_{M,N}$ and two disorder variables $\mu_{M,N}$ as defined by Kadanoff and Ceva.[10] We use the notation

$$\sigma[M_1,N_1;\ldots;M_n,N_n]\equiv\langle\sigma_{M_1,N_1}\sigma_{M_2,N_2}\cdots\sigma_{M_n,N_n}\rangle \tag{2}$$

and

$$\mu_{ij}[M_1,N_1;\ldots;M_n,N_n]\equiv\langle\sigma_{M_1,N_1}\cdots\mu_{M_i,N_i}\cdots\mu_{M_j,N_j}\cdots\sigma_{M_n,N_n}\rangle, \tag{3}$$

where in Eq. (3) the two disorder variables are associated with sites (M_i,N_i) and (M_j,N_j), the arguments of $\sigma[\cdots]$ and $\mu_{ij}[\cdots]$ will not be explicitly given except when needed, and μ_{ij} is defined to be antisymmetric in i and j.

For the interaction of Eq. (1) we introduce the notation $S_i(M,N)=\sinh[2E_i(M,N)/kT]$ for $i=1,2$ and we obtain the following results:

$$S_1(M_i,N_i)S_2(M_j,N_j)\{\sigma[\cdots]\sigma[M_i,N_i+1;\,M_j+1,N_j]-\sigma[M_i,N_i+1]\sigma[M_j+1,N_j]\}$$
$$=\mu_{ij}[\cdots]\mu_{ij}[M_i-1,N_i;M_j,N_j-1]-\mu_{ij}[M_i-1,N_i]\mu_{ij}[M_j,N_j-1] \tag{4a}$$

for $i \neq j$;

$$S_1(M_i,N_i)S_1(M_j,N_j)\{\sigma[\cdots]\sigma[M_i,N_i+1;\ M_j,N_j+1]-\sigma[M_i,N_i+1]\sigma[M_j,N_j+1]\}$$
$$=\mu_{ij}[M_i-1,N_i]\mu_{ij}[M_j-1,N_j]-\mu_{ij}[\cdots]\mu_{ij}[M_i-1,N_i;M_j-1,N_j] \tag{4b}$$

for $i \neq j$ and $(M_i,N_i)\neq(M_j,N_j)$;

$$S_2(M_i,N_i)S_2(M_j,N_j)\{\sigma[\cdots]\sigma[M_i+1,N_i;M_j+1,N_j]-\sigma[M_i+1,N_i]\sigma[M_j+1,N_j]\}$$
$$=\mu_{ij}[M_i,N_i-1]\mu_{ij}[M_j,N_j-1]-\mu_{ij}[\cdots]\mu_{ij}[M_i,N_i-1;M_j,N_j-1] \tag{4$\tilde{\text{b}}$}$$

for $i \neq j$ and $(M_i,N_i)\neq(M_j,N_j)$;

$$S_2(M_l,N_l)\{\sigma[\cdots]\mu_{ij}[M_l+1,N_l]-\sigma[M_l+1,N_l]\mu_{ij}[\cdots]\}$$
$$=\mu_{li}[\cdots]\mu_{lj}[M_l,N_l-1]-\mu_{li}[M_l,N_l-1]\mu_{lj}[\cdots] \tag{4c}$$

for $l\neq i$, $l \neq j$, and $i\neq j$; and

$$S_1(M_l,N_l)\{\sigma[\cdots]\mu_{ij}[M_l,N_l+1]-\sigma[M_l,N_l+1]\mu_{ij}[\cdots]\}$$
$$=-\mu_{li}[\cdots]\mu_{lj}[M_l-1,N_l]+\mu_{li}[M_l-1,N_l]\mu_{lj}[\cdots] \tag{4$\tilde{\text{c}}$}$$

for $l\neq i$, $l \neq j$, and $i\neq j$.

In addition, because of the arbitrariness of the placement of the string in the definition of the disorder variable[10] a relative minus sign is appended to some μ_{ij} wherever lattice sites on both sides of the string are involved.

These five equations cannot by themselves completely determine the correlation because no reference to T_c is yet included. The needed extra equation(s) will *not* be the same for all arbitrary lattices. We here specialize our attention to the translationally invariant Ising model where E_1 and E_2 are independent of M and N. In this case it is notationally convenient to absorb a factor of $(S_1S_2)^{1/2}$ into μ_{ij} and to define $\gamma_1=2z_2(1-z_1^2)$, $\gamma_2=2z_1(1-z_2^2)$, and $z_i=\tanh(E_i/kT)$, and rewrite Eqs. (4), respectively, as

$$\sigma[\cdots]\sigma[M_i,N_i+1;M_j+1,N_j]-\sigma[M_i,N_i+1]\sigma[M_j+1,N_j]$$
$$=\mu_{ij}[\cdots]\mu_{ij}[M_i-1,N_i;M_j,N_j-1]-\mu_{ij}[M_i-1,N_i]\mu_{ij}[M_j,N_j-1], \tag{5a}$$

$$\sigma[\cdots]\sigma[M_i,N_i+1;M_j,N_j+1]-\sigma[M_i,N_i+1]\sigma[M_j,N_j+1]$$
$$=(\gamma_1/\gamma_2)\{\mu_{ij}[M_i-1,N_i]\mu_{ij}[M_j-1,N_j]-\mu_{ij}[\cdots]\mu_{ij}[M_i-1,N_i;M_j-1,N_j]\}, \tag{5b}$$

$$\sigma[\cdots]\sigma[M_i+1,N_i;M_j+1,N_j]-\sigma[M_i+1,N_i]\sigma[M_j+1,N_j]$$
$$=(\gamma_2/\gamma_1)\{\mu_{ij}[M_i,N_i-1]\mu_{ij}[M_j,N_j-1]-\mu_{ij}[\cdots]\mu_{ij}[M_i,N_i-1;M_j,N_j-1]\}, \tag{5$\tilde{\text{b}}$}$$

$$\sigma[\cdots]\mu_{ij}[M_l+1,N_l]-\sigma[M_l+1,N_l]\mu_{ij}[\cdots]$$
$$=(\gamma_2/\gamma_1)^{1/2}\{\mu_{li}[\cdots]\mu_{lj}[M_l,N_l-1]-\mu_{li}[M_l,N_l-1]\mu_{lj}[\cdots]\}, \tag{5c}$$

and

$$\sigma[\cdots]\mu_{ij}[M_l,N_l+1]-\sigma[M_l,N_l+1]\mu_{ij}[\cdots]$$
$$=-(\gamma_1/\gamma_2)^{1/2}\{\mu_{li}[\cdots]\mu_{lj}[M_l-1,N_l]-\mu_{li}[M_l-1,N_l]\mu_{lj}[\cdots]\}. \tag{5$\tilde{\text{c}}$}$$

For this translationally invariant case we obtain the extra equation

$$(\gamma_2/\gamma_1)^{1/2}\{\sigma[M_i,N_i+1]\mu_{ij}[M_i,N_i-1]+\sigma[M_j,N_j+1]\mu_{ij}[M_j,N_j-1]\}$$
$$+(\gamma_1/\gamma_2)^{1/2}\{\sigma[M_i+1,N_i]\mu_{ij}[M_i-1,N_i]+\sigma[M_j+1,N_j]\mu_{ij}[M_j-1,N_j]\}$$
$$+\sum_{l\neq i,j}\{\mu_{li}[M_l,N_l-1]\mu_{lj}[M_l-1,N_l]-\mu_{li}[M_l-1,N_l]\mu_{lj}[M_l,N_l-1]\}$$
$$=2a(\gamma_1\gamma_2)^{-1/2}\sigma[\cdots]\mu_{ij}[\cdots], \tag{6}$$

where $a=(1+z_1^2)(1+z_2^2)$. When $n=2$, these equations reduce to the results of Ref. 4.

There are several properties of these equations which should be noted:

(i) Because of the arbitrariness inherent in specifying the relation of the order to disorder variables the equations remain valid if for any disorder variable the replacements $M_l \to M_l - 1$ or $N_l \to N_l - 1$, or both, are made. Thus, for example, from Eq. (4c) we obtain

$$S_2(M_l,N_l)\{\sigma[\cdots]\mu_{ij}[M_l+1,N_l;M_i-1,N_i]-\sigma[M_l+1,N_l]\mu_{ij}[M_i-1,N_i]\}$$
$$=\mu_{il}[M_i-1,N_i]\mu_{lj}[M_l,N_l-1]-\mu_{il}[M_l,N_l-1;M_i-1,N_i]\mu_{lj}[\cdots]. \tag{7}$$

(ii) There is also a symmetry if we interchange order and disorder variables. For example, if in Eq. (4b) we make the replacements $\sigma[\cdots]\to\mu_{jk}[\cdots]$, $\mu_{ij}[\cdots]\to\mu_{ik}[M_j+1,N_j+1]$, and $S_1(M_j,N_j)\to S_2^{-1}(M_j,N_j+1)$, we obtain

$$S_1(M_i,N_i)\{\mu_{jk}[\cdots]\mu_{jk}[M_i,N_i+1;M_j,N_j-1]-\mu_{jk}[M_i,N_i+1]\mu_{jk}[M_j,N_j-1]\}$$
$$=S_2(M_j,N_j)\{\mu_{ik}[\cdots]\mu_{ik}[M_i-1,N_i;M_j+1,N_j]-\mu_{ik}[M_i-1,N_i]\mu_{ik}[M_j+1,N_j]\}. \tag{8}$$

Equations (7) and (8) are consequences of Eqs. (4).

(iii) Correlations involving four or more disorder variables can be expressed as Pfaffians of correlations with only two disorder variables, e.g.,

$$\sigma\mu_{ijkl}=\mu_{ij}\mu_{kl}-\mu_{ik}\mu_{jl}+\mu_{il}\mu_{jk}. \tag{9}$$

Finally we take the scaling limit $T\to T_c$. Define

$$y=[2(a-\gamma_1-\gamma_2)\gamma_1^{-1}]^{1/2}M,\quad x=[2(a-\gamma_1-\gamma_2)\gamma_2^{-1}]^{1/2}N,\quad \bar\sigma=\mathfrak{M}^{-n}\sigma,\quad \bar\mu_{ij}=\mathfrak{M}^{-n}\mu_{ij},$$

where

$$\mathfrak{M}=|1-[\sinh(2E_1/kT)\sinh(2E_2/kT)]^{-2}|^{1/8}. \tag{10}$$

Then in the limit $a-\gamma_1-\gamma_2\to 0$ (where $\mathfrak{M}\to 0$) and $M,N\to\infty$, with x, y, $\bar\sigma$, and $\bar\mu_{ij}$ fixed, we find that Eqs. (5) become, respectively,

$$\bar\sigma\frac{\partial^2\bar\sigma}{\partial x_i\partial y_j}-\frac{\partial\bar\sigma}{\partial x_i}\frac{\partial\bar\sigma}{\partial y_j}=\bar\mu_{ij}\frac{\partial^2\bar\mu_{ij}}{\partial x_j\partial y_i}-\frac{\partial\bar\mu_{ij}}{\partial x_j}\frac{\partial\bar\mu_{ij}}{\partial y_i}, \tag{11a}$$

$$\bar\sigma\frac{\partial^2\bar\sigma}{\partial x_i\partial x_j}-\frac{\partial\bar\sigma}{\partial x_i}\frac{\partial\bar\sigma}{\partial x_j}=\frac{\partial\bar\mu_{ij}}{\partial y_i}\frac{\partial\bar\mu_{ij}}{\partial y_j}-\bar\mu_{ij}\frac{\partial^2\bar\mu_{ij}}{\partial y_i\partial y_j}, \tag{11b}$$

$$\bar\sigma\frac{\partial^2\bar\sigma}{\partial y_i\partial y_j}-\frac{\partial\bar\sigma}{\partial y_i}\frac{\partial\bar\sigma}{\partial y_j}=\frac{\partial\bar\mu_{ij}}{\partial x_i}\frac{\partial\bar\mu_{ij}}{\partial x_j}-\bar\mu_{ij}\frac{\partial^2\bar\mu_{ij}}{\partial x_i\partial x_j}, \tag{11$\tilde{\text{b}}$}$$

$$\bar\sigma\frac{\partial\bar\mu_{lj}}{\sigma y_l}-\frac{\partial\bar\sigma}{\partial y_l}\bar\mu_{ij}=-\mu_{il}\frac{\partial\bar\mu_{lj}}{\partial x_l}+\frac{\partial\bar\mu_{il}}{\partial x_l}\bar\mu_{lj}, \tag{11c}$$

and

$$\bar\sigma\frac{\partial\bar\mu_{lj}}{\partial x_l}-\frac{\partial\bar\sigma}{\partial x_l}\bar\mu_{ij}=\bar\mu_{il}\frac{\partial\bar\mu_{lj}}{\partial y_l}-\frac{\partial\bar\mu_{il}}{\partial y_l}\bar\mu_{lj}. \tag{11$\tilde{\text{c}}$}$$

Furthermore, Eq. (6) simplifies in this limit through the use of the above equations to

$$\tfrac{1}{2}\bar\sigma\left[\frac{\partial^2\bar\mu_{ij}}{\partial x_i^2}+\frac{\partial^2\bar\mu_{ij}}{\partial y_i^2}+\frac{\partial^2\bar\mu_{ij}}{\partial x_j^2}+\frac{\partial^2\bar\mu_{ij}}{\partial y_j^2}\right]+\tfrac{1}{2}\bar\mu_{ij}\left[\frac{\partial^2\bar\sigma}{\partial x_i^2}+\frac{\partial^2\bar\sigma}{\partial y_i^2}+\frac{\partial^2\bar\sigma}{\partial x_j^2}+\frac{\partial^2\bar\sigma}{\partial y_j^2}\right]$$
$$-\frac{\partial\bar\sigma}{\partial x_i}\frac{\partial\bar\mu_{ij}}{\partial x_i}-\frac{\partial\bar\sigma}{\partial y_i}\frac{\partial\bar\mu_{ij}}{\partial y_i}-\frac{\partial\bar\sigma}{\partial x_j}\frac{\partial\bar\mu_{ij}}{\partial x_j}-\frac{\partial\bar\sigma}{\partial y_j}\frac{\partial\bar\mu_{ij}}{\partial y_j}=\bar\sigma\bar\mu_{ij}. \tag{11d}$$

It can be shown that Eqs. (11) are equivalent to those of Sato, Miwa, and Jimbo.[9]

Two separate derivations may be given for these equations, corresponding to the different derivations of the two-point-function results of Ref. 4. Equations (4) on the inhomogeneous lattice (finite or infinite planar lattice) may be derived by applying the generalized Wick theorem[11] after expressing the correlations in terms of Clifford-algebra operators.[1] The violations in the equations are obtained by invoking the ordering operator needed to go from the transfer-matrix description back to the Ising-model correlations.

On the translationally invariant lattice, Eqs. (5) may also be derived by starting with the dispersion expansion integrals[12] and calculating suitable first differences. In this derivation we may generalize the functions by introducing an appropriate n-dimensional matrix Λ of constants into the integrals (corresponding to the scalar λ of Ref. 4). For all Λ the integrals satisfy the equations but the violations will only be correct if Λ has the Ising value.

For both methods of derivation an additional calculation explicitly involving translational invariance is needed to obtain Eq. (6).

It is also worth making explicit that in none of our derivations is the concept of monodromy used.

Detailed derivation of the results of this paper will be presented elsewhere.

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