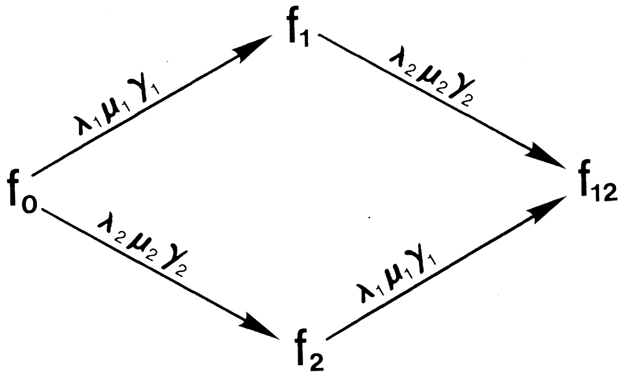


FIG. 1. Commutability of the Bäcklund transform.

which is different but seemingly similar to Eq. (9) is known.[15] In the framework of the IST method, I can generalize the customary Zakharov-Shabat scheme which generates usual soliton and "oscillating" soliton solutions to include the present similarity-type "ripplon" solutions as well.[16]

Finally, I note that the present decay-mode solution does not have proper limit in $\alpha \to 0$ (one-dimensional KdV limit) at the present form. In this respect, it seems that much still remains to be studied about decay modes in the most general form.

I thank Dr. M. J. Ablowitz for discussions. This work is partially supported by the U. S. Air Force Office of Scientific Research under Contract No. 78-3674B and by the Office of Naval Research under Contract No. N00014-76-C-0867.

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# Sine-Gordon Solitons Do Not Behave Like Newtonian Particles

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(Received 3 December 1980)

It is shown that the acceleration of the sine-Gordon solitons by a constant torque increases as $t^2$ for small times $t$. For larger times, the velocity behaves like $\tanh t^3$. Thus the soliton dynamical behavior is not Newtonian. Analytical arguments are given based on the linear perturbation theory which generalize this conclusion to any weak external perturbation of sine-Gordon solitons.

PACS numbers: 03.20.+i

The sine-Gordon (SG) equation including a constant external field $\chi$ is written in dimensionless form

$$\frac{\partial u}{\partial t^2} - \frac{\partial u}{\partial x^2} + \sin u = \chi. \qquad (1)$$

This equation is related to many problems of great physical interest.[1,2] In the present Letter, we are only concerned with the solitary-wave solution of (1).[1-3] A very simple and suggestive illustration of the basic properties of this solution is found in the SG pendulum chain system perturbed by the application of a constant torque $\chi$ to each pendulum.[4,5] We first restrict ourselves

 

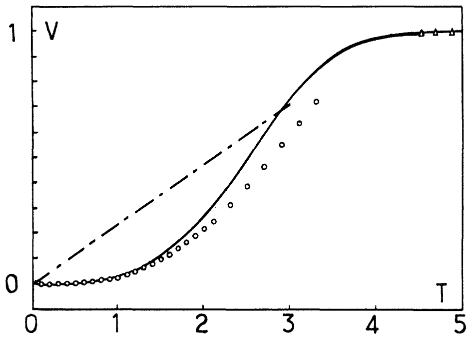


FIG. 1. Soliton velocity $V$ vs time $t$ for $\Gamma=0$ and $\chi=0.3$. Circles and triangles, numerical values obtained from (2) and from direct measurements of $(\partial u/\partial x)_{max}$ ($V<1$ and $V \lesssim 1$, respectively); dash-dotted line, Newton's linear law $V=\frac{1}{4}\pi\chi t$; solid line, the theoretical formula (7) with a weak time dependence of $\gamma$ obtained numerically.

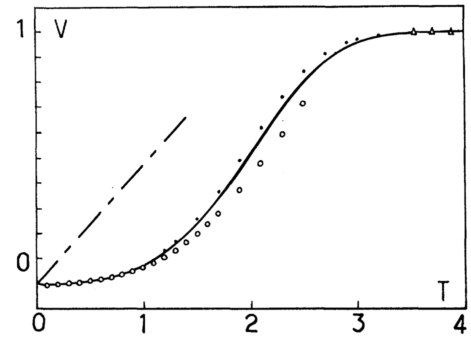


FIG. 2. $\Gamma=0$; $\chi=0.6$; the dots show formula (7) where $\gamma\equiv 1$.

in this Letter to the case of a SG soliton accelerated by a constant field without dissipation. We numerically show that such a soliton does not obey the *a priori* expected Newton dynamics[3] and give some analytical arguments to explain this result. Then we show that the addition in (1) of a small linear "viscous damping" $\Gamma\partial u/\partial t$ does not qualitatively change the conclusions of this paper concerning the non-Newtonian acceleration of the solitary wave. We conclude by pointing out the generality of this new effect: We analytically show that the non-Newtonian nature of the soliton dynamics does not depend on the special form of the given perturbation (which may be localized in space or not).

In Figs. 1–3, we plot the evolution of the soliton velocity $V$ versus time $t$ for increasing values of the applied field $\chi$. The initial condition is in all cases a static ($V=0$) antikink SG solution. This choice does not reduce the generality of the results since a Lorentz transformation to the soliton rest frame is always possible.[3] We have numerically obtained (by the leap-frog method with three points in time and seven points in space and using a classical Simpson method for the integrals) the velocity $V$ according to

$$V(t)=\frac{\Delta}{\Delta t}\left(\frac{\sum_i x_i(\Delta u/\Delta x)_i\,\Delta x}{\sum_i(\Delta u/\Delta x)_i\,\Delta x}\right). \qquad (2)$$

The increments $\Delta x$ and $\Delta t$ have been respectively chosen to be 0.25 and 0.1. The accuracy of the numerical results has been checked in the case of $\chi=0$ by comparing them with the dynamics of an exact soliton solution. The error was less than 10% (the same order of magnitude as for the deviation between numerical and theoretical curves in Figs. 1–4).

Definition (2) means that we measure the velocity of the solitary wave at its center of mass. It is acceptable only for $V<1$ (circled line in Figs. 1–4), since spatial oscillations growing in the tail of the solitary wave as $V$ increases become important when $V\to V_{lim}=1$ and significantly perturbs the kernel $\partial u/\partial x$ in (2).[8] For further measurements of $V$, we located the abscissas of the soliton spatial derivative maxima $(\partial u/\partial x)_{max}$ and plotted the corresponding velocity versus time in Figs. 1–4 (small triangles).

The results are quite surprising. They show, even for small values of $\chi$, a definite deviation from the Newtonian kinetic law $V(t)=\frac{1}{4}\pi\chi t$.[2,3] Moreover, the smaller the time $t$, the more important this deviation. This is the opposite of what is expected from a classical particle accel-

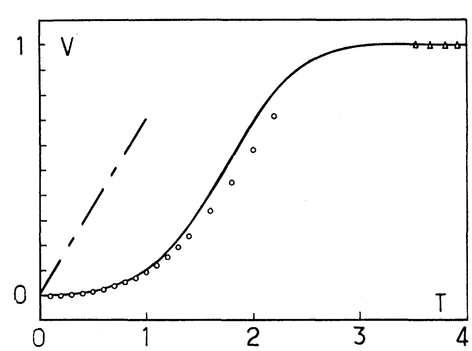


FIG. 3. $\Gamma=0$; $\chi=0.9$. Note that $\chi>\chi_{threshold}=0.7246$ which implies that $u_\infty(t)\to\infty$ (Refs. 6 and 7). Nevertheless, for small $t$, $\sin u_\infty \sim u_\infty$ and (5) remains valid.

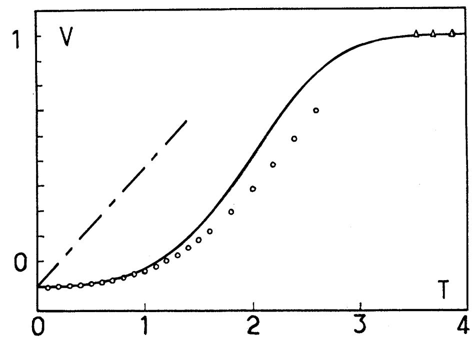


FIG. 4. $\Gamma=0.1$; $\chi=0.6$; then $V_{\rm lim}=0.98$.

erated by a constant force. As a matter of fact, there is a progressive acceleration proportional to $t^2$ and, later on, an inflection point. We note that these anomalous effects did not appear in the work of Nakajima and Onodera[9] because they adopted a too small time scale in their Fig. 2. Finally, we point out that the same basic features of this non-Newtonian kinetics still exist when a "damping effect" $\Gamma\partial u/\partial t$ is added to the left-hand side of (1) (see Fig. 4).[10]

We use the technique of the "perturbed potential well" and refer to Refs. 11 and 12 for a detailed description of this method. For $x\rightarrow\pm\infty$ Eq. (1) reduces to

$$d^2u_\infty/dt^2+\sin u_\infty=\chi, \tag{3}$$

where $u_\infty(t)=u(\pm\infty,t)$. We look for a solution of the form $u(x,t)=f(z)+u_\infty(t)$ where $z=\gamma(t)[x-\int_0^t V(t')dt']$ and $f(z)\approx 4\tan^{-1}\exp(-z)$. For $t\rightarrow 0$, the above *Ansatz* is appropriate since the perturbation function $\psi=u-f$ determined by (1) is $\frac{1}{2}\chi t^2$. The substitution of this *Ansatz* in (1), together with the requirement that $f$ must remain regular at both wings ($z\rightarrow\pm\infty$), leads to the so-called force equation,[11,12] which reduces to (3) for $z\rightarrow\pm\infty$. This force equation may be integrated and we obtain the "energy equation" which leads, at $z\rightarrow\infty$, to

$$\gamma(t)V(t)=\tfrac{1}{4}\pi[\chi t-du_\infty/dt], \tag{4}$$

since we assume as initial condition an exact static antikink solution. As this soliton starts to move, we have $\sin u_\infty\sim u_\infty$ in (3), and therefore $u_\infty(t)\sim\chi[1-\cos t]$. Equation (4) then implies

$$V(t)\sim\pi\chi t^3/24\gamma(t). \tag{5}$$

This formula, which fits very well the numerical data plotted on Figs. 1–3 for $t\gtrsim 0$, describes a definite deviation from Newton's law $V(t)=\frac{1}{4}\pi\chi t$,[3] since it leads to a time-dependent acceleration (in $t^2$), even for small values of the velocity $V$. We note that such an effect avoids the physical paradox lying in the soliton inertial mass $m_i=4/\pi$ being different from the soliton rest mass $m_r=8$.[1]

The further kinetic behavior of the antikink for a larger $t$ may be understood as follows: At a given value of time $t$, we introduce the tangent Galilean frame moving with a velocity $V(t)$ with respect to the laboratory frame.[2] Then, for sake of simplicity in the calculations, we make two classical assumptions[2,3]: (i) The deviation of the soliton trajectory with respect to the trajectory of the tangent frame is small enough to allow the use of formula (5) in which the laboratory time $t$ has to be replaced by its Lorentz-transformed value $\tau=(1-V^2)^{1/2}t$. This approximation is rather strong since it implies that $\sin u_\infty\sim u_\infty$ even for a large $t$. (ii) The time variation of the wave number $\gamma$ remains small and $\gamma$ itself does not significantly differ from the Lorentz factor $(1-V^2)^{-1/2}$.

We are aware of the contradiction introduced by these approximations, since they imply that we consider small values of $V$ compared with unity and at the same time we assume that the antikink still behaves as a classical relativistic particle. Nevertheless, we numerically check that they lead to acceptable values of $V$ (even when it becomes rather large). We have $V(t+dt)=[V(t)+\mathcal{V}(\tau+d\tau)][1+V(t)\mathcal{V}(\tau+d\tau)]^{-1}$, where $\mathcal{V}$ is the relative velocity of the antikink with respect to the tangent frame. Since $\mathcal{V}(\tau)=0$, we obtain, in first order in $\mathcal{V}/V$,

$$(1-V^2)^{-1}dV/dt\approx d(\gamma^{-1}\mathcal{V})/d\tau. \tag{6}$$

Integrating Eq. (6) and adopting for $\mathcal{V}(\tau)$ expression (5), we obtain the "relativistic" generalization of the solitary-wave kinetics (5):

$$V(t)\approx\tanh(\pi\chi t^3/24\gamma^5). \tag{7}$$

When a small damping ($0<\Gamma\ll 1$) is present in Eq. (1), we obtain, by proceeding as above, the corresponding soliton kinetics:

$$V(t)\approx\tanh[(\pi\chi t^3/24\gamma^5)(1-\tfrac{1}{2}\Gamma t)]. \tag{8}$$

In summary, we show on Figs. 1–4 that formulas (7) and (8) agree fairly well with the numerical plots of $V(t)$. Their deviation with respect to the numerical values may be reduced when the wave number $\gamma$ is allowed to increase slowly with time, according to the actual numerical results (see Fig. 2).

This non-Newtonian soliton acceleration is a quite general result, valid in particular for the whole time scale where the soliton motion remains nonrelativistic. To demonstrate this important result in terms of the pioneering linearized perturbation theory,[3] we perturb the SG equation about its kink solution and solve the resulting linear inhomogeneous equation for the perturbation function $\psi(z,\tau)$ in the soliton rest frame by expanding it in the complete set of eigenfunctions $\{f_b, f_k\}$ of the Schrödinger operator with potential $1-2\,\mathrm{sech}^2 z$. Calling $g(t,x)$ the general form of a weak ($g \ll 1$) perturbation at the right-hand side of the SG equation, we obtain the following kinetic equations for the amplitudes $\psi_b(\tau) = (\psi, f_b)$ and $\psi_k(\tau) = (\psi, f_k)$ where the brackets mean the scalar product in the set $\{f_b, f_k\}$[3]:

$$d^2\psi_b/d\tau^2 = \int_{-\infty}^{\infty} f_b(z) g(\tau,z) dz\,; \qquad d^2\psi_k/d\tau^2 + \omega_k^2\psi_k = \int_{-\infty}^{\infty} f_k^*(z') g(\tau,z') dz' \tag{9}$$

($\omega_k$ is the eigenvalue corresponding to $f_k$); therefore

$$\frac{\partial^2\psi(z,\tau)}{\partial\tau^2} = \frac{d^2\psi_b}{d\tau^2} f_b(z) + \int_{-\infty}^{\infty} \frac{d^2\psi_k}{d\tau^2} f_k(z) dk. \tag{10}$$

The substitution of expressions (9) in (10) and the use of the completeness relation $f_b(z_1)f_b(z_2) + \int_{-\infty}^{\infty} dk f_k^*(z_1) f_k^*(z_2) = \delta(z_1 - z_2)$ *cancels the total amplitude of the translation mode* $f_b$ [once $\psi$ is assumed small enough at $t \gtrsim 0$ in order to neglect $\omega_k^2\psi_k$ in Eq. (9)]. Since this amplitude measures the soliton displacement due to the perturbation,[3] *we conclude that the soliton does not move at all within the frame of this linear perturbation theory* [i.e., for small time values $t \gtrsim 0$].[3] Actually, it does, but according to a higher-order acceleration process, as in the case of a constant torque $\chi$, where the soliton undergoes an acceleration proportional to $\chi t^2$ instead of $\chi$. This higher-order effect is because of the account of the neglected term $\omega_k^2\psi_k$ in (9). The perturbation function $\psi$ measured in the soliton rest frame is therefore only static and yields from (10) $\partial^2\psi/\partial\tau^2 = g(\tau,z)$ as expected since we assumed $t \gtrsim 0$.

The above result generalizes the conclusion of the present paper concerning the non-Newtonian nature of the SG soliton dynamics to all kinds of weak external perturbation, localized in space or not. It does not mean that the localized wave never behaves like a particle. We only claim that this particle is not Newtonian. In particular, when a final–steady-state regime is possible (for instance, in the above case of a weakly damped SG soliton accelerated by a constant force $\chi$), the perturbed soliton may well behave like a *relativistic* particle of velocity $V_\infty = \frac{1}{4}\pi\chi/\Gamma$, while the way (8) the soliton adjusts itself in order to satisfy the new boundary conditions imposed by the perturbation and reach this final "equilibrium" state is definitely *not* Newtonian [see also Ref. (10)]. When $\Gamma = 0$, this remark holds in a less accurate sense, since the profile of the "particle" is never stationary in the soliton rest frame, though the soliton reaches its asymptotic (relativistic) velocity equal to unity.

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