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Nonlinear Intermediate Long-Wave Equation: Analysis and Method of Solution

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A physically interesting nonlinear singular integro-differential equation which is an intermediary between the Korteweg–deVries and Benjamin-Ono equations is considered via the inverse-scattering transform. Novel aspects of the theory and limits to the Benjamin-Ono equation are discussed.

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Recent studies have shown that the equation

$$u_t+\delta^{-1}u_x+2uu_x+T(u_{xx})=0, \qquad (1)$$

where $(Tu)(x)=⨍_{-\infty}^{\infty} dy\, u(y)\coth[\pi(y-x)/2\delta]/2\delta$ ($⨍_{-\infty}^{\infty}$ represents the principal-value integral), is of mathematical and physical interest. Physically it represents long waves in a stratified fluid of finite depth characterized by the parameter δ.[1,2] Depending on δ we get the Korteweg–de Vries (KdV) equation as $\delta\to 0$ (shallow-water limit),

$$u_t+2uu_x+(\delta/3)u_{xxx}=0, \qquad (2)$$

and the Benjamin-Ono (BO) equation as $\delta\to\infty$ (deep-water limit),

$$u_t+2uu_x+H(u_{xx})=0, \qquad (3)$$

where $(Hu)(x)=\pi^{-1}⨍_{-\infty}^{\infty} dy\, u(y)/(y-x)$ (Hilbert transform). Hence Eq. (1) is an intermediary equation between these two very interesting nonlinear evolution equations. Hereafter we shall refer to (1) as the intermediate long-wave (ILW) equation. Mathematically speaking, Eq. (1) has soliton solution,[3,5] a Bäcklund transformation, and a novel type of linear scattering problem.[5,6] In this Letter we shall do the following:

(a) Relate Eq. (1) directly to a linear Gel'fand-Levitan integral equation which has N-soliton solutions.

(b) Discuss how to deal with this new scattering problem. We show in what sense the above Gel'fand-Levitan equation can be derived from analytical considerations of suitable scattering data.

(c) We shall also briefly discuss the limiting case of the BO equation for which there has also been considerable study (see, for example, Refs. 7–12) regarding solitons, Bäcklund transformations, and linear scattering problems.

We begin with point (a). The operator T defined below Eq. (1) immediately suggests a splitting of the function $u(x)$ into appropriate analytic functions. Namely, if we call $U(z)=(Tu)(z)$, $\mathrm{Im}z\neq 0$, then the boundary values on $z=x$, x real, satisfy $U^{\pm}(x)=(Tu)(x)\pm iu(x)$. Here $U^{\pm}(x)$ are the boundary values of functions analytic in the horizontal strip between $\mathrm{Im}z=0$ and $\mathrm{Im}z=\pm 2\delta$, and are periodically extended vertically. Moreover, periodicity requires that $U^-(x)=U^+(x+2i\delta)$. It is convenient to define $g(x)\equiv -U^+(x+i\delta)/2$ [here $g(z)$ is analytic in the strip $-\delta<\mathrm{Im}z<\delta$], whereupon the splitting takes the form $u(x)=i[g(x-i\delta)-g(x+i\delta)]$, $(Tu)(x)=-[g(x-i\delta)+g(x+i\delta)]$. Hence

(1) takes the form

$$i(g^+-g^-)_t+i\delta^{-1}(g^+-g^-)_x-2(g^+-g^-)(g^+-g^-)_x-(g_{xx}{}^+ +g_{xx}{}^-)=0, \tag{4}$$

where $g^\pm(x)\equiv g(x\mp i\delta)=-U^\pm(x)/2$.

Consider the following linear Gel'fand-Levitan integral equation:

$$K(x,y)+F(x,y)+\int_x^\infty K(x,s)F(s,y)\,ds=0,\quad \text{for } y>x. \tag{5}$$

Following the basic idea of Zakharov and Shabat,[13] we introduce linear operators on F, such that

$$L_1F=(i\partial_x+\tfrac{1}{2}\delta^{-1})F^+(x,y)+(i\partial_y-\tfrac{1}{2}\delta^{-1})F^-(x,y)=0, \tag{6a}$$

$$L_2F=(i\partial_t+\partial_x{}^2-\partial_y{}^2)F(x,y)=0, \tag{6b}$$

where $F^\pm=F(x\mp i\delta,\,y\mp i\delta)$. Then, direct calculation shows that $K(x,y)$ must satisfy

$$[i\partial_x+\tfrac{1}{2}\delta^{-1}+iK^+(x,x)-iK^-(x,x)]K^+(x,y)+(i\partial_y-\tfrac{1}{2}\delta^{-1})K^-(x,y)=0, \tag{7a}$$

$$[i\partial_t+\partial_x{}^2-\partial_y{}^2+2\partial_xK(x,x)]K(x,y)=0. \tag{7b}$$

Compatibility between Eqs. (7a) and (7b) gives us Eq. (4) with $g^\pm(x)=K^\pm(x,x)\equiv K(x\mp i\delta,x\mp i\delta)$. The N-soliton solutions to Eq. (1) can now be readily constructed (in the usual manner) by assuming exponential solutions for F; i.e., $F(x,y)=\sum_{l=1}^N C_l(t)\exp(i\zeta_{-l}x+i\zeta_{+l}y)$, where $\zeta_{\pm l}=i\kappa_l\pm[\kappa_l\cot(2\kappa_l\delta)-\frac{1}{2}\delta^{-1}]$, $\kappa_l>0$ and $C_l(t)=C_l(0)\exp[-4\kappa_l(\kappa_l\cot 2\kappa_l\delta-\frac{1}{2}\delta^{-1})t]$. A one-soliton solution is given by $u=2\kappa_1\sin(2\kappa_1\delta)/\{\cos(2\kappa_1\delta)+\cosh[2\kappa_1(x-x_0(t))]\}$, where $x_0(t)=(2\kappa_1)^{-1}\ln[C_1(t)/2\kappa_1]$.

We now pass to point (b). As discussed in Ref. 6, the linear scattering problem obeys (with some changes in notation)

$$i\psi_x{}^+ +(u-\lambda)\psi^+=\mu\psi^-, \tag{8a}$$

$$i\psi_t{}^\pm+2i(\lambda+\tfrac{1}{2}\delta^{-1})\psi_x{}^\pm+\psi_{xx}{}^\pm+[\mp iu_x-T(u_x)+\nu]\psi^\pm=0, \tag{8b}$$

where $\lambda=k\coth(2k\delta)$, $\mu=k\,\mathrm{csch}(2k\delta)$, $\nu=k^2-2k(\lambda+\frac{1}{2}\delta^{-1})$, and here $\psi^\pm(x)$ represent the boundary values of functions analytic in the horizontal strips between Im $z=0$ and Im $z=\pm2\delta$, and periodically extended. As mentioned earlier this implies $\psi^-(x)=\psi^+(x+2i\delta)$. We note that this condition immediately leads to $T(\psi^+-\psi^-)=i(\psi^++\psi^-)$ which is required in Ref. 6, and that compatibility of Eqs. (8a) and (8b) yields Eq. (1). In order to analyze the scattering problem, it is convenient to define a new function, $W^+(x,k)\equiv\psi^+(x,k)\exp[ik(x-i\delta)]$, whereupon the scattering problem becomes

$$\mathcal{L}W=iW_x{}^+ +(\zeta_++\tfrac{1}{2}\delta^{-1})(W^+-W^-)=-uW^+ \tag{9}$$

with $W^-(x)=W^+(x+2i\delta)$, $\zeta_\pm(k)=k\pm[k\coth(2k\delta)-\frac{1}{2}\delta^{-1}]$ (We shall need the definition of ζ_- subsequently.) Now we define specific Jost functions for real k: $M^+(x;k)\to1$, as $x\to-\infty$, and $N^+(x;k)\to\exp[2ik(x-i\delta)]$, $\bar N^+(x;k)\to1$, as $x\to+\infty$. Each of these functions can be shown to satisfy an integral equation. For this purpose, we introduce the notion of a Green function satisfying $\mathcal{L}G(x,y;k)=-\delta(x-y)$ [$\mathcal{L}$ defined by Eq. (9).] Then

$$M^+(x;k)=1+\int_{-\infty}^\infty G_1{}^+(x,y;k)u(y)M^+(y;k)\,dy, \tag{10a}$$

$$N^+(x;k)=\exp[2ik(x-i\delta)]+\int_{-\infty}^\infty G_2{}^+(x,y;k)u(y)N^+(y;k)\,dy, \tag{10b}$$

$$\bar N^+(x;k)=1+\int_{-\infty}^\infty G_2{}^+(x,y;k)u(y)\bar N^+(y;k)\,dy, \tag{10c}$$

where

$$G_{1,2}{}^+(x,y;k)=\tfrac{1}{2}\pi\int_{C_{1,2}}\hat G^+(p;k)\exp[ip(x-y)]\,dp, \tag{11}$$

and $\hat G^+(p;k)=\{p-(\zeta_++\frac{1}{2}\delta^{-1})[1-\exp(-2\delta p)]\}^{-1}$. $\hat G^+(p;k)$ has poles at $p_0=0$, $p=2\zeta_+{}^{-1}(\zeta_+(k))$. We note $\zeta_+{}^{-1}(\cdots)$ is a multivalued function and we have an infinite number of poles p for which we shall define $p_{-1}=2k$ and $p_n,\bar p_n$ ($n\geq1$) such that $(2n-1)\pi/2\delta<\mathrm{Im}\,p_n<(2n+3)\pi/2\delta$ and similarly for $-\mathrm{Im}\,\bar p_n$. Moreover double poles occur at special values of $\zeta_+(k)$ satisfying $k=0$ and $p=\zeta_+(k)$. We call these values $\{\zeta_+{}^{(i)},\bar\zeta_+{}^{(i)}\}_{i=1}^\infty$ [$\mathrm{Im}\,\zeta_+{}^{(i)}>0$, $\mathrm{Im}\,\bar\zeta_+{}^{(i)}<0$]. The contours C_1, C_2 are taken to be the lines $\mathrm{Re}\,p-i0$ and $\mathrm{Re}\,p+i0$, respectively (this is necessary in order to preserve the boundary conditions). It is important to remark that Eqs. (10a)–(10c) are Fredholm-type integral equations, unlike the usual case of the Schrödinger equation where the Jost functions satisfy Volterra equations. In addition we note that, by using residue calculus, Eqs. (10a)–(10c) can be represented in an explicit manner useful for the proof of existence and analyticity of the solution (convergence of Neumann series). From Eqs. (10a)–(10c) one can establish the following [assuming $u(x)$ decays rapidly as $|x|\to\infty$]:

(i) M^+, N^+, and $\overline{N}^+$ have convergent Neumann series in certain regions of ζ_+ plane for given δ and $\max|u|$ chosen small enough.

(ii) In the ζ_+ plane, p_n ($n\geq 1$) has a logarithmic branch point at $\zeta_+=-1/2\delta$ and square-root branch points at $\zeta_+^{(n)}$ and $\zeta_+^{(n+1)}$.

(iii) Despite (ii), $M^+(x;k)$ and $\overline{N}^+(x;k)$ are analytic in the upper and lower half ζ_+ plane, respectively, whenever the Neumann series converges in this region. Moreover, as $|\zeta_+|\to\infty$, $M^+,\overline{N}^+\to 1+O(1/\zeta_+)$. Here we note that k is a multivalued function of ζ_+, and we are required to define an appropriate branch in k plane. For the functions M^+ and $\overline{N}^+$, our principal branch is that one containing the real k axis, and which has $\mathrm{Im}\,k\gtrless 0$ corresponding to $\mathrm{Im}\,\zeta_+\gtrless 0$. There is a branch point at $\zeta_+=-1/2\delta$ and a branch cut from $\zeta_+=-1/2\delta$ to $\zeta_+=-\infty$.

By virtue of the fact $G_1^+(x,y;k)-G_2^+(x,y;k)=(2i\delta\zeta_+)^{-1}-\exp[2ik(x-y)]/2i\delta\zeta_-$, we have a relation among $M^+(x;k)$, $N^+(x;k)$, and $\overline{N}^+(x;k)$ for real k [i.e., $\zeta_+>-1/(2\delta)$],

$$M^+(x;k)=a(k)\overline{N}^+(x;k)+b(k)N^+(x;k), \qquad (12)$$

where $a(k)=1+[\int_{-\infty}^{\infty}u(y)M^+(y;k)dy]/2i\delta\zeta_+$, $b(k)=-\{\int_{-\infty}^{\infty}u(y)M^+(y;k)\exp[-2ik(y-i\delta)]dy\}/2i\delta\zeta_-$. Hence $a(k)$ takes on the same analyticity as $M^+(x;k)$, and as $|\zeta_+|\to\infty$, $a(k)\to 1$. On the other hand, for ζ_++i0 with $\zeta_+<-1/2\delta$ and real (i.e., k is in the upper half plane at the edge of the principal branch), we have a relation $G_1^+(x,y;k)-G_2^+(x,y;k^*)=1/2i\delta\zeta_+$ [note $\zeta_+(k+i0)=\zeta_+(k^*-i0)$, k^* complex conjugate of k], which yields

$$M^+(x;k)=a(k)\overline{N}^+(x;k^*). \qquad (13)$$

The bound states [as $x\to\infty$, $M^+(x;k)\to 0$] are defined by $a(k_l)=0$, $M^+(x;k_l)=b_lN^+(x;k_l)$, for $\mathrm{Im}\,k_l>0$ ($l=1,2,\ldots,N$). The scattering data are now given by $S=\{a(k),b(k),\{k_l,b_l\}_{l=1}^{N}\}$. We have found that $a(k)$ has only simple zeros and they lie on the imaginary k axis, i.e., $k_l=i\kappa_l$. From Eqs. (10b–10c) and consistent with our analyticity requirements, we assume, for $\overline{N}^+$ and N^+, the triangular representations

$$\overline{N}^+(x;k)=1+\int_x^{\infty}ds\,K^+(x,s)\exp[i\zeta_+(x-s)],\ \text{for}\ \mathrm{Im}\zeta_+<0, \qquad (14a)$$

$$N^+(x;k)=\exp[2ik(x-i\delta)]+\int_x^{\infty}ds\,K^+(x,s)\exp[i\zeta_+(x-s)+2ik(s-i\delta)], \qquad (14b)$$

where $K^+(x,s)$ satisfies Eq. (7) and $K^+(x,s)\to 0$ as $s\to\infty$. Inverse scattering formulas are obtained as follows: Divide Eq. (12) and Eq. (13) by $a(k)$ and operate with $(1/2\pi)\int_{-\infty}^{\infty}d\zeta_+\exp[i\zeta_+(y-x)]$ (i.e. Fourier transform) for $y>x$. Then using Eq. (14), we obtain the linear Gel'fand-Levitan integral Eq. (5) with

$$F(x,y)=\tfrac{1}{2}\pi\int_{-1/2\delta}^{\infty}d\zeta_+\frac{b(k)}{a(k)}\exp(i\zeta_-x+i\zeta_+y)+\sum_{l=1}^{N}C_l\exp(i\zeta_{-l}x+i\zeta_{+l}y), \qquad (15)$$

where $C_l=-ib_l/\dot{a}_l$ and $\dot{a}_l=[\partial a/\partial\zeta_+]_{\zeta_+=\zeta_{+l}}$. From Eq. (8b), the time dependence of the scattering data is given by $a(k,t)=a(k,0)$, $b(k,t)=b(k,0)\exp[-4\times ik(\lambda+\frac{1}{2}\delta^{-1})t]$, for real k, $b_l(t)=b_l(0)\exp[4\kappa_l(\lambda_l+\frac{1}{2}\delta^{-1})t]$. We expect the Gel'fand-Levitan equation is valid when the Neumann series expansions of Eq. (10) converge. For fixed $\max|u(x,0)|$, when $\delta\to\infty$ (the BO limit), this will not hold and new singularities due to the Fredholm nature of Eq. (10) may have to be taken into account. We briefly mention this later.

We now pass on to point (c). Our basic philosophy regarding the BO equation is to obtain information by taking the limit process $\delta\to\infty$. First of all if we simply take $\delta\to\infty$, then for real k, $\zeta_+=2k\theta(k)$ and $\hat{G}^+(p;k)=(p-2k)^{-1}\theta(p)$, where $\hat{G}^+(p;k)$ is defined below Eq. (11) and $\theta(\cdots)$ is the usual Heaviside step function. Similarly from the fact that $G^-(x,y;k)=G^+(x+2i\delta,y;k)$ we have $\hat{G}^-(p;k)=\hat{G}^+(p;k)\exp(-2\delta p)\to\theta(-p)/2k$. These formulas suggest a natural splitting of the δ function and the Green function. Hence from these results, we may deduce the split equations for the eigenfunctions,

$$iM_x^+(x;k)+2k[M^+(x;k)-1]=\hat{P}^+(uM^+)(x;k), \qquad (16a)$$

$$M^-(x;k)=1+(2k)^{-1}\hat{P}^-(uM^+)(x;k), \qquad (16b)$$

where $\hat{P}^{\pm}=\frac{1}{2}(1\mp iH)$ are the usual projection operators. It is also worthwhile noting that the eigenvalue problem with $k<0$ ($\delta\to\infty$) in Eq. (16) corresponds to what happens to $\zeta_+<-1/2\delta$ for finite

δ. Moreover we have found a solution of the *homogeneous* equation for $N^+(x;k)$ with some $k<0$ (this seems to be related to BO solitons). In this regard, we note that one can actually compute certain eigenfunctions of the scattering problem for the BO equation. We shall use the scattering problem [from Eq. (9) with $\delta\to\infty$]:

$$iW_x^{\ +}+(u+2k)W^+=2kW^- . \tag{17}$$

Let us consider $u(x)=2\nu/(x^2+1)=i\nu[1/(x+i)-1/(x-i)]$ as an example. There is a natural way to split the eigenvalue problem Eq. (17). Namely multiply by $(x-i)$ and require both sides to be an entire function. For the case of bound states (solitons) we take $2k(x-i)W^-=1$. Then the solution for W^+ can be found to be

$$W^+=-i\left(\frac{x-i}{x+i}\right)^\nu\int_{-\infty}^x\left(\frac{y+i}{y-i}\right)^\nu\frac{\exp[2ik(x-y)]}{y-i}dy, \tag{18}$$

requiring $W^+\to 0$ as $|x|\to\infty$. This implies immediately that

$$D_\nu(k)\equiv\int_{-\infty}^{\infty}\left(\frac{y+i}{y-i}\right)^\nu\frac{\exp(-2iky)}{y-i}\,dy=0 \tag{19}$$

is the condition which determines the discrete eigenvalues. For $\nu=n=$integer, Eq. (19) is the Laguerre polynomial of degree n, i.e., $D_n(k)=L_n(-4k)=0$, for $k<0$. Hence, for $\nu=n$, there are n-real distinct eigenvalues, e.g., for $n=1$, $k_1=-1/4$; for $n=2$, $k_{1,2}=-(2\pm\sqrt{2})/4$, etc. Moreover, this condition corresponds to the requirement that W^+ is, in fact, analytic in the upper half plane. Thus we expect to find n solitons when $\nu=n$ (in agreement with Ref. 14). The situation with $\nu\neq$integer is more difficult. Nevertheless we found that Eq. (18) has n eigenvalues for ν in the range $n-1<\nu\leq n$ ($n=1,2,\dots$). We also remark that when $\nu=1$ the eigenfunction $W^+(x;k)$ satisfies the homogeneous equation of Eq. (10b) with $G_2^{\ +}$ given by $G_2^{\ +}(x,y;k)=(2\pi)^{-1}\int_0^\infty dp(p-2k)^{-1}\exp[ip(x-y)]$ for $k=k_1<0$.

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