Solvable q-State Models in Lattice Statistics and Quantum Field Theory

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The commutation of transfer matrices of q-state lattice models is studied and solutions which generalize both the $q=2$ ferroelectric models and the special $q=3$ models of Stroganov are found. For cases which are continuously connected to a shift operator, associated commuting Hamiltonians are given and free energies calculated.

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Recently there has been much interest in the SU(N) generalizations[1,2] of the Thirring model.[3] These studies[2] are made directly in the continuum and are solved by using Bethe's hypothesis[4] in a manner similar to that used by Yang[5] and by Sutherland[6] for the multicomponent δ-function gases.

To specify these models fully, a renormalization procedure must be defined. For the (one-component) Thirring model, this was most beautifully carried out by Luther[7] and by Lüscher,[8] who put the field theory on a spatial lattice and related it to the X-Y-Z model solved by Baxter.[9] In particular, this renormalization is powerful enough to demonstrate spontaneous mass generation out of the (bare) massless model.[10] For the multicomponent case, the corresponding renormalization on the lattice has not been carried out. Indeed the work of Sutherland[11] indicates that a straightforward generalization of the Heisenberg-Ising model related to the field theory will not in general be solvable on the lattice. Accordingly, it is the purpose of this Letter to initiate the study of the lattice renormalization of these field theories by determining the q-state lattice models which may be solved by the techniques of commuting transfer matrices.

For $q=2$, a complete study of commuting transfer matrices exists for the eight-vertex model.[12] I generalize this to a model on a $M\times M$ square lattice with periodic boundary conditions by allowing q states (or colors) on each bond and by requiring conservation of color at each vertex. Then, in the notation of Fig. 1 the Boltzmann weight $W=e^{-\epsilon/kT}$ of a vertex specified by $\lambda,\lambda';\alpha,\alpha'$ is

$$R^{\lambda\lambda'}(\alpha\alpha')=W_{\alpha\lambda}{}^{d}\delta_{\lambda\lambda'}\delta_{\alpha\alpha'}+W_{\alpha\alpha'}{}^{r}\delta_{\alpha\lambda}\delta_{\alpha'\lambda'}+W_{\alpha\alpha'}{}^{l}\delta_{\alpha\lambda'}\delta_{\alpha'\lambda}, \tag{1}$$

where by definition $W_{\alpha\alpha}{}^{l}=W_{\alpha\alpha}{}^{r}=0$. There are in general $3q^2-2q$ arbitrary vertex weights.

The transfer matrix is constructed from $R(\alpha,\alpha')$ considered as a $q\times q$ matrix as

$$T_{\alpha_1\alpha_2\cdots\alpha_N|\alpha_1'\alpha_2'\cdots\alpha_N'}=\mathrm{Tr}[R(\alpha_1,\alpha_1')R(\alpha_2,\alpha_2')\cdots R(\alpha_N,\alpha_N')] \tag{2}$$

and the lattice partition function is

$$Z=\mathrm{Tr}(T^M). \tag{3}$$

Note that the left shift operator

$$T_L=\delta_{\alpha_1\alpha_2'}\delta_{\alpha_2\alpha_3'}\cdots\delta_{\alpha_N\alpha_1'} \tag{4}$$

is given by Eq. (2) with $W_{\alpha\beta}{}^{d}=\delta_{\alpha\beta}$, $W_{\alpha\beta}{}^{r}=0$, and $W_{\alpha\beta}{}^{l}=1-\delta_{\alpha\beta}$.

I search for sets of weights, $\{W\}$ and $\{\overline{W}\}$, such that

$$[T,\overline{T}]=0 \tag{5}$$

and by Eq. (2) it is sufficient to construct a nonsingular $q^2\times q^2$ matrix $X_{\mu\mu'}^{\lambda\lambda'}$, such that, for arbitrary $\lambda,\lambda';\mu,\mu';\alpha,\alpha'$,

$$\sum_{\alpha'',\lambda'',\mu''}X_{\mu\mu''}^{\lambda\lambda''}R^{\lambda''\lambda'}(\alpha,\alpha'')\overline{R}^{\mu''\mu'}(\alpha'',\alpha')=\sum_{\alpha'',\lambda'',\mu''}\overline{R}^{\mu\mu''}(\alpha,\alpha'')R^{\lambda\lambda''}(\alpha'',\alpha')X_{\mu''\mu'}^{\lambda''\lambda'}. \tag{6}$$

Thus there are q^6 equations to satisfy.

The matrix $X_{\mu\mu'}^{\lambda\lambda'}$ may be chosen to have the

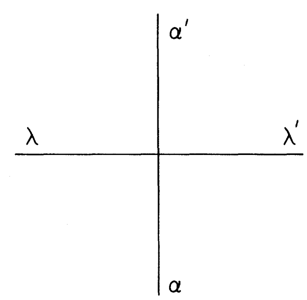

FIG. 1. Labeling of the bonds of a single vertex used to write the single-site transfer matrix of the text.

same form, Eq. (1), as R; then we find the following two special types of solutions.

(I) Generalized six-vertex models, where

$$W_{\rho\sigma}{}^{l}=0, \quad W_{\rho\sigma}{}^{r}=W_{\sigma\rho}{}^{r}, \tag{7}$$

or the alternate set with $l \leftrightarrow r$; there are two cases: We have case (IA)

$$\frac{(W_{\rho\sigma}{}^{r})^2}{W_{\rho\sigma}{}^{d}W_{\sigma\rho}{}^{d}}=\frac{(\overline{W}_{\rho\sigma}{}^{r})^2}{\overline{W}_{\rho\sigma}{}^{d}\overline{W}_{\sigma\rho}{}^{d}}, \tag{8}$$

$$\frac{W_{\rho\sigma}{}^{d}}{W_{\rho\rho}{}^{d}}=\frac{\overline{W}_{\rho\sigma}{}^{d}}{\overline{W}_{\rho\rho}{}^{d}}, \tag{9}$$

or the alternate set with $l \leftrightarrow r$. These constitute $3q(q-1)$ restrictions on the W's; therefore this is a q-parameter family of commuting transfer matrices, of which one parameter is a trivial overall scale. For case (IB) we define the index-independent quantities K and $\overline{K}$ by

$$\frac{(W_{\rho\sigma}{}^{r})^2}{W_{\rho\sigma}{}^{d}W_{\sigma\rho}{}^{d}}=K^2, \quad \frac{(\overline{W}_{\rho\sigma}{}^{r})^2}{\overline{W}_{\rho\sigma}{}^{d}\overline{W}_{\sigma\rho}{}^{d}}=\overline{K}^2, \tag{10}$$

or the alternate set with $r \leftrightarrow l$; then there is commutation if

$$W_{\rho\sigma}{}^{d}/W_{\rho\gamma}{}^{d}=\overline{W}_{\rho\sigma}{}^{d}/\overline{W}_{\rho\gamma}{}^{d}=c_{\rho\sigma\gamma} \tag{11}$$

and

$$W_{\rho\rho}{}^{d}(W_{\sigma\gamma}{}^{d}W_{\gamma\sigma}{}^{d})^{1/2}(W_{\rho\sigma}{}^{d}W_{\sigma\rho}{}^{d}W_{\rho\gamma}{}^{d}W_{\gamma\rho}{}^{d})^{-1/2}=K\pm 1, \tag{12}$$

where the $\pm$ sign may depend on γ, ρ, and σ; and similar equations hold with W's $\to \overline{W}$'s and $K\to\overline{K}$. For a given set of $c_{\rho\sigma\gamma}$'s, restricted by equations of the form $c_{\rho\sigma\tau}c_{\sigma\rho\gamma}c_{\gamma\tau\sigma}c_{\tau\gamma\rho}=(K\pm 1)(K\pm 1)^{-1}$ obtained from Eqs. (11) and (12), we have a commuting family with $q+1$ arbitrary parameters for general q.

These cases generalize the $q=2$ ferroelectric model[13] solved by Bethe's hypothesis.

(II) Index-independent models, where (with $\rho \neq \sigma$)

$$W_{\rho\rho}{}^{d}=W^{d}, \quad W_{\rho\sigma}{}^{d}=W^{0}, \quad W_{\rho\sigma}{}^{r}=W^{r}, \quad W_{\rho\sigma}{}^{l}=W^{l}; \tag{13}$$

there are several cases: For case (IIA) we have

$$W^0=0; \tag{14}$$

this corresponds to the q-state generalization of the three-state models of Stroganov,[14] of which there are two subcases: We have subcase (IIA1)

$$W^d-W^r-W^l=0 \tag{15}$$

and the corresponding barred equation; and subcase (IIA2)

$$W^r=W^d(W^d-W^l)[W^d+(q-2)W^l]^{-1} \tag{16}$$

and the corresponding barred equation. In both subcases there are two free parameters, of which one is the trivial scale. For case (IIB) we have

$$W^0\neq 0; \tag{17}$$

here also there are two subcases: We have subcase (IIB1) when

$$W^r\neq 0 \text{ and } W^l\neq 0; \tag{18}$$

then there is commutation if

$$W^d=W^r+W^l\pm W^0 \tag{19}$$

and

$$2W^0(W^r+W^l)=\mp(q-2)W^rW^l. \tag{20}$$

This is a two-parameter family. We have subcase (IIB2) when

$$W^rW^l=0; \tag{21}$$

then there is commutation if, in addition, Eq. (19) holds. This is also a two-parameter family and may be obtained from case (IB) by directly making the specialization Eq. (13).

For the remainder of this note I concentrate on type-II solutions where there is a two-parameter family which contains the left (or right) shift operator, Eq. (4), as a special case. For these cases we may find a commuting Hamiltonian by taking the logarithmic derivative of the transfer matrix with respect to the one nontrivial parameter at the shift point. We may also find one-variable recursion relations for the lattice free energy by the method of Stroganov,[14] which may then be explicitly solved. The cases with $(q+1)$- or q-parameter families must be dealt with by different techniques and will be treated elsewhere. The results are summarized as follows:

Subcase (IIA1).—The associated Hamiltonian is

$$\mathcal{H}=\sum_{J=1}^{N}\Big[\sum_{\rho<\sigma}\tfrac{1}{2}(\Lambda_J^{\rho\sigma x}\Lambda_{J+1}^{\rho\sigma x}-\Lambda_J^{\rho\sigma y}\Lambda_{J+1}^{\rho\sigma y})+\sum_{\rho=1}^{q}\Lambda_J^{\rho d}\Lambda_{J+1}^{\rho d}\Big], \tag{22}$$

where the Λ's are a basis set of $q \times q$ matrices with elements

$$(\Lambda^{\rho d})_{ij} = \delta_{i\rho}\delta_{j\rho}, \quad (\Lambda^{\rho\sigma x})_{ij} = \delta_{i\rho}\delta_{j\sigma} + \delta_{i\sigma}\delta_{j\rho}, \quad (\Lambda^{\rho\sigma y})_{ij} = -i(\delta_{i\rho}\delta_{j\sigma} - \delta_{i\sigma}\delta_{j\rho}). \tag{23}$$

The recursion relation is

$$f(b) f\left(\frac{1+(q-1)b}{1+(q-2)b}\right) = \frac{[1+(q-1)b](1-b)}{1+(q-2)b}, \tag{24}$$

where $b = W^r/W^d$. The solution is

$$f(b) = \frac{u^2 x^{1/2}}{(1+x)} (q-2)^{-1/2} R\left(\frac{x}{u^2}\middle| u^4\right) R\left(\frac{1}{u^2 x}\middle| u^4\right), \tag{25}$$

where

$$R(x\,|\,\alpha) = \prod_{k=1}^{\infty} \left(\frac{1 - x\alpha^{1-2k}}{1 - x\alpha^{-2k}}\right), \tag{26}$$

$$x = -\frac{b + u^{-1}(q-2)^{-1/2}}{b - u(q-2)^{-1/2}}, \tag{27}$$

$$u = \tfrac{1}{2}[(q-2)^{1/2} + (q+2)^{1/2}]. \tag{28}$$

Subcase (IIA2).—The Hamiltonian is

$$\mathcal{H} = \sum_{J=1}^{N} \left[\sum_{\rho<\sigma} \tfrac{1}{2}(\Lambda_J^{\rho\sigma x} \Lambda_{J+1}^{\rho\sigma x} - \Lambda_J^{\rho\sigma y} \Lambda_{J+1}^{\rho\sigma y}) + \sum_{\rho=1}^{q} (q-1) \Lambda_J^{\rho d} \Lambda_{J+1}^{\rho d} \right]. \tag{29}$$

From the recursion relation

$$f(b) f(1+(q-1)b) = \frac{[1+(q-1)b](1-b)}{1+(q-2)b}, \tag{30}$$

we obtain

$$f(b) = \frac{(q-1)}{(q-2)} R\left(\frac{(q-2)}{(q-1)} x \,\middle|\, (q-1)\right) R\left(\frac{1}{(q-2)x} \,\middle|\, (q-1)\right), \tag{31}$$

$$x = b + (q-2)^{-1}. \tag{32}$$

Subcase (IIB1).—We find the Hamiltonian functions

$$\mathcal{H}^{\pm} = \sum_{J=1}^{N} \left(\pm(q-4)(q-2)^{-1} \sum_{\rho=1}^{q} \Lambda_J^{\rho d} \Lambda_{J+1}^{\rho d} + \sum_{\rho<\sigma} \left\{ [\tfrac{1}{2} \mp (q-2)^{-1}] \Lambda_J^{\rho\sigma x} \Lambda_{J+1}^{\rho\sigma x} + [\tfrac{1}{2} \pm (q-2)^{-1}] \Lambda_J^{\rho\sigma y} \Lambda_{J+1}^{\rho\sigma y} \right\} \right). \tag{33}$$

The recursion relation is

$$f(c) f(c \pm \delta) = c[(c \pm \delta) - (c \pm \delta)^{-1}], \tag{34}$$

where

$$\delta = (q-2)/2, \tag{35}$$

$$c = W^0/W^r, \tag{36}$$

giving

$$f(c) = c\, \frac{\Gamma^2(\frac{1}{2} \pm c/2\delta)\, \Gamma(1 \pm (c-1)/2\delta)\, \Gamma(1 \pm (c+1)/2\delta)}{\Gamma^2(1 \pm c/2\delta)\, \Gamma(\frac{1}{2} \pm (c-1)/2\delta)\, \Gamma(\frac{1}{2} \pm (c+1)/2\delta)}. \tag{37}$$

Subcase (IIB2).—The Hamiltonian is

$$\mathcal{H} = \sum_{J=1}^{N} \left[\sum_{\rho<\sigma} \tfrac{1}{2}(\Lambda_J^{\rho\sigma x} \Lambda_{J+1}^{\rho\sigma x} + \Lambda_J^{\rho\sigma y} \Lambda_{J+1}^{\rho\sigma y}) \pm \sum_{\rho=1}^{q} \Lambda_J^{\rho d} \Lambda_{J+1}^{\rho d} \right]. \tag{38}$$

The recursion relation is

$$f(b) = \frac{[1+(q-1)b](1-b)}{1+(q-2)b} f\left(\frac{b}{(1+qb)}\right), \tag{39}$$

where

$$b = W^r/W^d, \tag{40}$$

with solution

$$f(b) = \frac{\Gamma(1/qb)\, \Gamma([1+(q-2)b]/qb)}{\Gamma((1-b)/qb)\, \Gamma([1+(q-1)b]/qb)}. \tag{41}$$

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