I get $r_{max} < 2.25 \times 10^{15} \xi^{1/5}$.

Thus whatever the cooling method, a machine operating at $r \gtrsim 10^{15}$ must overheat by hundreds of degrees Kelvin and destroy itself (a conservative time scale is 10^{-6} sec). Therefore, 10^{15} operations/sec is a firm upper bound on the speed of an ideal digital computer. Any realistic machine would fall short of this by orders of magnitude.

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[a]On sabbatical leave from the Ben Gurion University of the Negev, Beersheva, Israel.

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Power Spectra of Strange Attractors

B. A. Huberman and Albert B. Zisook[a]
Xerox Palo Alto Research Center, Palo Alto, California 94304

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It is shown that, for systems which enter chaos through period doubling bifurcations, the integrated noise power spectrum in the chaotic regime behaves as $N(r) = N_0(r - r_c)^{\sigma}$, with $\sigma = 1.5247\ldots$. Furthermore, the existence of a new universal constant which describes the scaling behavior of the average bandwidth in the strange attractor is reported. These results are directly applicable to experiments probing the onset of turbulence in physical systems.

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A number of physical systems, such as stressed fluids, high-temperature plasmas, and Josephson junctions have been observed to undergo a transition into a turbulent regime characterized by broadband noise in the power spectra. A possible explanation for these phenomena is that the phase trajectories for the complete nonlinear many-body problem enter a low-dimensional region of phase space containing a strange attractor. A strange attractor is a region in phase space such that nearby trajectories must enter it but once inside they diverge from each other. Hence we arrive at a description of turbulence involving only very few degrees of freedom. The effectively stochastic motion which these few degrees of freedom undergo gives rise to the observed noise in the power spectra. One common route into this turbulent regime is a universal cascade of period doubling bifurcations which occur as some control parameter is varied.[1-3] This cascade can be easily understood when, through the construction of return maps associated with the Poincaré maps, the dynamical system is mapped onto one-dimensional (1D) recursion relations which possess the same bifurcation structure.[4]

Recently, it has been shown that once in the chaotic regime, the Lyapunov exponent, which measures the rate of divergence of nearby trajectories, behaves very much like the order parameter of a phase transition near the critical point, i.e., it obeys a universal scaling law.[5] This development allows, in principle, for the application of techniques developed in the study of critical phenomena to the onset of turbulence in these nonlinear systems.

Appealing as these ideas might be, they suffer from the fact that one cannot directly measure Lyapunov exponents or discern the topology of attractors in experiments dealing with physical systems. We therefore believe that what is needed is a theory of the power spectra associated with strange attractors above the chaotic threshold.

This paper presents the main results of such a theory. In particular, we show that, for systems which enter the chaotic regime through a sequence of period doubling bifurcations, the integrated noise power spectrum in the chaotic regime, $N(r)$, behaves as

$$N(r)=N_0(r-r_c)^{\sigma}, \tag{1}$$

where $\sigma=1.5247\ldots$. Furthermore, we report the existence of a new universal constant, β, associated with 1D maps displaying bifurcation cascades, which describes the scaling behavior of the average bandwidth in the strange attractor. Our results, which are in very good agreement with numerical simulations, should be directly applicable to experiments probing the onset of turbulence in a variety of physical systems.

It has been shown that the return map for dynamical systems displaying period doubling bifurcations, such as anharmonic systems,[4] corresponds to simple recursion relations of the form $x_{n+1}=f(x_n)$, where $f(x)$ has a single parabolic maximum. Because of the universality of the bifurcation structure of these 1D maps, we may consider the simplest one defined on the inverval $[0, 1]$ (with $0 \leq r \leq 4$), i.e.,

$$f(x)=rx(1-x). \tag{2}$$

For such a map it is well known[6] that, as the control parameter r is increased, a cascade of period doubling bifurcations takes place for $r=r_n$, $n=1,2,3,\ldots$, until a value $r=r_c$ is reached beyond which chaotic behavior ensues. Beyond r_c, a reverse set of bifurcations, or band mergings, occurs for the values $r=\tilde{r}_n$, $n=1,2,3,\ldots$.[7] Moreover, at $r=\tilde{r}_n$ the action of $f^{(2^n)}(x)$ on any fixed band is completely chaotic, its invariant measure being a scaled down version of that of the map $f(x)$ with $r=4$.

If the iterates generated by the map are denoted by $x_k=f^{(k)}(x_0)$, we can represent the sequence x_k (for $r=\tilde{r}_n$) by

$$x_k=\sum_j A_j \exp[i(\omega_j)k]+n(k) \tag{3}$$

with

$$A_j=\lim_{N\to\infty}(1/N)\sum_{n=1}^{N}\exp[-i(\omega_j)n]x_n, \tag{4}$$

where the frequencies ω_j are integral multiples of $2\pi/2^n$ and $n(k)$ is the noise term generated by the deterministic map. Since $n(k)$ is a scaled down version of the noise obtained from the map $f(x)$ with $r=4$ we can write[7]

$$\lim_{T\to\infty}\left[T^{-1}\sum_{t=1}^{T}n(t+k)n(t)\right]=W^2\delta(k) \tag{5}$$

with W^2 a constant proportional to the average of the square of the width of one of the 2^n bands. The power spectrum $G(\omega)$ is simply calculated from Eq. (3). We thus obtain

$$G(\omega)=|x(\omega)|^2=|\sum_j A_j\delta(\omega-\omega_j)+n(\omega)|^2. \tag{6}$$

Therefore, the power spectrum in the chaotic regime will consist of a set of instrumentally narrow peaks sitting on top of a broadband noise background. The existence of these δ-function peaks in the chaotic regime are a reflection of the time-translation–invariant property of the driving term in the original dynamical system.[8]

In order to proceed further in our analysis of the broadband noise we first establish a new universal property of 1D recursion relations with period doubling bifurcations. We have found that for highly bifurcated orbits in the chaotic regime (or $r=\tilde{r}_n$ with n large) the root-mean-square bandwidth obeys the scaling relation

$$W_n=W_0\beta^{-n} \tag{7}$$

with $\beta=3.2375\ldots$.[9] We believe that this new universal constant is independent of the constants $\alpha=2.5029\ldots$ and $\delta=4.6692\ldots$ introduced by Feigenbaum.[10] It should be noted that Eq. (7) also describes the scaling of the average spacing of the most highly bifurcated pairs in the periodic regime with $r=r_n$.

The experimental quantity of interest in the chaotic regime is the integrated power density $N(r)$, which is given by

$$N(r)=\int d\omega\,|n(\omega)|^2, \tag{8}$$

where we have subtracted the periodic structure given by the δ functions. The range of integration is $0\leq\omega\leq\omega_d$, where ω_d is the driving frequency in the original system. For discrete maps, the driving frequency is 2π. To determine the scaling behavior, we use an argument similar to that used by Huberman and Rudnick in their study of Lyapunov exponents.[5] First we notice that, as r goes from $\tilde{r}_n$ to $\tilde{r}_{n+1}$ in the reverse bifurcation sequence, Eqs. (5), (7), and (8) imply that

$$N(\tilde{r}_{n+1})=N(\tilde{r}_n)\beta^{-2}. \tag{9}$$

Furthermore, since for large n, the value of r

$=\tilde{r}_n$ for which the nth reverse bifurcation, or band merging, takes place behaves as[7,10]

$$\tilde{r}_n = r_c + \text{const}\,\delta^{-n}, \tag{10}$$

we obtain, from Eqs. (9) and (10),

$$N(r) = \text{const}(r - r_c)^{\sigma}, \tag{11}$$

with σ a universal exponent which is given by $\sigma = 2\ln(\beta)/\ln(\delta) = 1.5247\ldots$. Therefore, the total noise power, which in turn defines an effective noise temperature, T_{eff}, obeys a scaling law near the chaotic threshold.

In order to test these predictions we have measured the power spectrum associated with the 1D map of Eq. (2). To within the accuracy of our calculations we obtained $\sigma = 1.527 \pm 0.005$, in excellent agreement with the theoretical value. We thus believe that experiments in systems displaying period doubling bifurcations will observe the scaling behavior given by Eq. (11).

Our arguments rest on the existence of the special points $\tilde{r}_n$ at which the bands are completely mixing. Since one may question the validity of our prediction for $r \neq \tilde{r}_n$ because of the existence of highly bifurcated shallow periodic attractors, it is important to consider the effects of external noise. As has been recently shown[11] the addition of small amounts of external noise to the dynamical system results in the disappearance of these shallow attractors. Therefore, experimental determinations of our scaling predictions will unavoidably interpolate smoothly between the special reverse bifurcation points, leading to a simple determination of the power-law behavior predicted by Eq. (11).

In summary, we have shown that (a) the power spectrum of the chaotic phase of period doubling systems consists of δ functions and broadband noise, (b) the average noise scales near onset with a universal power law, and (c) our theoretically determined exponent agrees with our numerical experiments.

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[a]Permanent address: James Frank Institute, The University of Chicago, 5640 S. Ellis Ave., Chicago, Ill. 60637.

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