the order of 35% in transfer reactions.

The present paper shows evidence in transfer reactions for an important breaking of the supersymmetry selection rules proposed by Iachello,[1,3] with at least a 35% breaking of the supersymmetry scheme itself. In view of the interest of the problem, it should stimulate other experimental and theoretical works to understand further the limits of validity of the supersymmetry model.

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Relationship between the Bohr Collective Hamiltonian and the Interacting-Boson Model

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The version of the interacting-boson model (IBM) which treats neutrons and protons symmetrically can be converted into a Bohr Hamiltonian description in three steps: (1) Mapping the IBM pairing bosons onto a quadrupole particle-hole–type boson under a generalized Holstein-Primakoff transformation; (2) introduction of pseudocanonical coordinates and momenta; (3) expansion in powers of the pseudomomenta, valid for large particle numbers. The limiting symmetries of the IBM emerge directly from a study of the potential-energy surface.

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In view of the impressive achievments of the interacting-boson model (IBM) in describing nuclear collective motion at low excitation energies,[1] there has been mounting interest in understanding as precisely as possible its connection with the established phenomenology, the Bohr collec-

tive Hamiltonian,[2,3] of which it is a generalization. In preliminary reports, a method based on the use of coherent states[4] has been applied to the computation of the collective potential-energy surface[5] and more elaborately (in the guise of the method of generator coordinates[6,7]) to the computation of a full collective Hamiltonian.[8] (The direct comparison of this result with a Bohr Hamiltonian is problematical because of the absence of unitary equivalence of the two descriptions.) Previously, Moshinsky[9] had shown how the IBM Hamiltonian could be diagonalized in a representation based on the Bohr picture. Closest in spirit to the present work are discussions[10,11] in which the relevance of Eqs. (5)–(8) (below) to the problem at hand has been appreciated, but the further consequences which flow from it have not been drawn.

The purpose of the present note is to describe and implement a method which maps a standard IBM Hamiltonian into a Hamiltonian associated with an orthonormal basis which describes only the degrees of freedom (quadrupole oscillator) that occur in the Bohr picture. The approach followed is related to ideas suggested some time ago by one of the authors[12,13] within the framework of a simplified model. A subsequent generalization by Okubo[14] will be the starting point of the present work. The basic idea can be exposed by reference to a standard IBM Hamiltonian, expressed in terms of two bosons, the s boson, describing a pair of nucleons coupled to angular momentum 0, and the d_μ boson, $\mu = 1, \ldots, 5$, describing a coupling to angular momentum 2,

$$H = \epsilon \sum_\mu d_\mu^\dagger d_\mu - \kappa \sum_\mu Q_\mu Q_{-\mu} (-1)^\mu , \tag{1}$$

$$Q_\mu = s^\dagger \tilde{d}_\mu + d_\mu^\dagger s + \chi (d^\dagger d)_\mu^{(2)} , \tag{2}$$

$$\tilde{d}_\mu = (-1)^\mu d_{-\mu} . \tag{3}$$

The Hamiltonian (1) contains three parameters, ϵ describing the relative excitation energy of a d pair compared with an s pair, κ, the overall strength of the quadrupole-quadrupole force, and χ, the relative importance of the two kinds of basic quadrupole that define the general quadrupole operator (2).

The Hamiltonian (1) conserves the total number of bosons, N, where

$$N = s^\dagger s + \sum_\mu d_\mu^\dagger d_\mu , \tag{4}$$

and N is considered to be half the number of "active fermions." The goal of the program under consideration is to replace the six degrees of freedom of the IBM by a Hamiltonian containing five quadrupole degrees of freedom and the value of N. It is understood that the equivalent Bohr-Mottelson picture emerges in the limit as N tends to infinity. Thus, if we can display a transformed Hamiltonian which admits an expansion in inverse powers of N, our aim will have been essentially achieved. We proceed to do the task.

Generalized Holstein-Primakoff Mapping.—The 35 bilinear combinations of s, d, and their Hermitian conjugates which are linearly independent of (4), but commute with it, are the generators of SU(6). Our problem is greatly simplified by the circumstance that we require only a very special representation of this Lie algebra, namely the one completely characterized by a single quantum number, the value of N. As Okubo[14] has shown,[15] this representation can be constructed in terms of a new quadrupole boson, b_μ, by means of the formulas,

$$d_\mu^\dagger d_\nu = b_\mu^\dagger b_\nu , \tag{5}$$

$$d_\mu^\dagger s = b_\mu^\dagger (N-h)^{1/2} , \tag{6}$$

$$s^\dagger d_\mu = (N-h)^{1/2} b_\mu , \tag{7}$$

$$s^\dagger s = N - h , \tag{8}$$

where

$$h = \sum_\mu b_\mu^\dagger b_\mu \tag{9}$$

is clearly the number of b bosons. [A similar reduction from n to $n-1$ bosons can be carried out for SU(n). Only for $n=2$ does it yield all the representations, in this case reducing to the famous Holstein-Primakoff representation[16] of angular momentum.]

It may help the reader to consider the following interpretation of Eqs. (5)–(8): In the IBM, the basis for studying a nucleus with N bosons is obtained by applying to the "vacuum" various products $(s^+)^{n_s}(d^+)^{n_d}$ such that $n_s + n_d = N$. An equivalent basis is obtained by applying $(b^+)^n$ to the state $(s^+)^N|\text{vac}\rangle$, where $n \leq N$. The b boson is thus to be considered as a multipole rather than a pairing boson, and for this reason is the object which we ultimately seek to associate with the Bohr picture.

The substitution of (5)–(8) into H yields a first form of transformed Hamiltonian, which we do not record. In consequence of the square roots in (6) and (7) it is not a polynomial in the creation and annihilation operators. One might be tempted to expand in powers of h/N, but this ex-

pansion can be justified only in the limit of weak interaction among the bosons, which coincides with the SU(5) or vibrational limit of the IBM.[17] Since we wish to develop a theory that is also valid for the other special limits of interest, the O(6)[18] and SU(3)[19] limits, we must proceed with greater caution.

Limit of Large Particle Numbers.—In the examples studied previously,[12-14] the passage to the large-N limit was achieved by replacing the creation and annihilation operators by action-angle variables. In the meantime, we have recognized that the same ends can be achieved most simply by the introduction of canonical coordinates and momenta (strictly pseudocanonical coordinates[20]) by means of the standard formulas,

$$b_\mu{}^\dagger = (x_\mu - ip_\mu{}^\dagger)/\sqrt{2}\,, \tag{10}$$

$$b_\mu = (x_\mu{}^\dagger + ip_\mu)/\sqrt{2}\,, \tag{11}$$

and

$$x_\mu{}^\dagger = \tilde{x}_\mu\,, \quad p_\mu{}^\dagger = \tilde{p}_\mu\,, \tag{12}$$

$$[x_\mu, p_\nu] = i\delta_{\mu\nu} = [x_\mu{}^\dagger, p_\nu{}^\dagger]. \tag{13}$$

Below we shall first exhibit and then manipulate the Hamiltonian which results from the substitution of (10) and (11) into Eq. (1). First, however, we describe the physical basis for the steps to be taken. The argument which follows is well known.[12]

When dealing with the oscillatory regime, small vibrations about spherical equilibrium, the commutators (13) are fulfilled by matrix elements which, in order of magnitude, satisfy $x \sim p \sim 1$. On the other hand, when the system undergoes a phase transition to a deformed shape, as we shall see in detail below, $x \sim N^{1/2}$, so that in order to satisfy (13), we have $p \sim N^{-1/2}$. This suggests that in dealing with the radicals that occur in (6) and (7), we write (matrix-multiplication notation)

$$h = b^\dagger b = \tfrac{1}{2}(\tilde{x}x + \tilde{p}p + 5)\,, \tag{14}$$

and note that if we lump the factor $\tilde{x}x$ with the factor N and expand only in terms of the remaining quantities, $\tilde{p}p + 5$, then successive terms will certainly diminish by reciprocal powers of N. Though this expansion is justified for large N for all regimes of the parameters and is the only one generally valid, further expansion of the coordinate dependence is justified in the spherical domain.

The successive substitutions and expansion that we have described entail only the most elementary of algebraic manipulations. We therefore pass immediately to the final Hamiltonian which we wish to consider. For this we require some definitions and a change of variables. The latter involves a convenient rescaling

$$x \to N^{1/2}x\,, \quad p \to N^{-1/2}p\,. \tag{15}$$

We shall also encounter the following tensors of rank 0 and 2:

$$\tilde{x}x \equiv \beta^2\,, \tag{16}$$

$$(x \otimes x)_\mu{}^{(2)} \equiv \Xi_\mu\,, \tag{17}$$

$$\tilde{x}\Xi \equiv -(\tfrac{2}{7})^{1/2}\beta\cos 3\gamma\,, \tag{18}$$

$$(p \otimes p)^{(0)} = \tilde{p}p \equiv \Pi\,, \tag{19}$$

$$(p \otimes p)_\mu{}^{(2)} \equiv \Theta_\mu\,. \tag{20}$$

We exhibit a dimensionless Hamiltonian,

$$\mathcal{H} \equiv H/N\epsilon\,, \tag{21}$$

expressed in terms of a dimensionless coupling constant

$$F \equiv \kappa N/\epsilon\,, \tag{22}$$

namely

$$\mathcal{H} = \mathcal{T} + \mathfrak{U}\,, \tag{23}$$

where $\mathfrak{U}$ is the potential energy,[21]

$$\mathfrak{U} = \mathfrak{U}^{(0)} + N^{-1}\mathfrak{U}^{(1)} + O(N^{-2})\,, \tag{24}$$

$$\mathfrak{U}^{(0)} = (\tfrac{1}{2} - 2F)\beta^2 + F\chi[\tfrac{4}{7}(1 - \tfrac{1}{2}\beta^2)]^{1/2}\beta^3\cos 3\gamma + F[1 - (\chi^2/14)]\beta^4, \tag{25}$$

$$\mathfrak{U}^{(1)} = -\tfrac{1}{2}F\beta^2 + (\tfrac{9}{7})^{1/2}F\chi[1 - \tfrac{1}{2}\beta^2]^{-1/2}\beta^3\cos 3\gamma\,, \tag{26}$$

and $\mathcal{T}$ is the kinetic energy calculated to the lowest nonvanishing order only,

$$\mathcal{T}N^2 = \tfrac{1}{2}\Pi + \tfrac{1}{2}F\{\beta^2, \Pi\} - (\tfrac{1}{8}\sqrt{7})F\chi\{\beta^3\cos 3\gamma, \{(1 - \tfrac{1}{2}\beta^2)^{-1/2}, \Pi\}\} - (8)^{-1/2}F\chi\{(1 - \tfrac{1}{2}\beta^2)^{1/2}, \{x_\mu, \Theta_\mu\}\} - \tfrac{1}{4}F\chi^2\{\Xi_\mu, \Theta_\mu\}\,. \tag{27}$$

The curly brackets in (27) are anticommutators.

To understand fully the significance of the result contained in (21)–(27), we emphasize that if indeed

x and p, as redefined through (15), are each of order unity, then we have succeeded in developing the Hamiltonian in reciprocal powers of N. In the limit as $N \to \infty$, the energy per particle is determined completely by the function $\mathfrak{u}^{(0)}$, the so-called potential-energy surface. In this case the model yields a finite energy per particle. Reverting to the spherical or weak-coupling limit, we have instead that $\mathfrak{u}^{(0)}$ and $\mathcal{T}$ are the same order of magnitude, both of order N^{-1}. Thus, since the energy per particle tends to zero with increasing particle number, the energy per particle is still given correctly in this case by the potential energy alone (since it also gives zero). We conclude that it suffices to study the latter function to determine the various regimes or phases enjoyed by the model under consideration. However, before carrying through this discussion, we must change the definition of the variable β in order to be able to compare with Refs. 4 and 8. The variable β defined by Eq. (16) cannot exceed $\sqrt{2}$ in value; the variable employed in Refs. 4 and 8, which has the conventional range $0 \to \infty$, is related to ours by the equation

$$\beta = \sqrt{2}\,\beta'/(1+\beta'^2)^{1/2}, \tag{28}$$

so that in the sequel we shall study the function (dropping primes)

$$(1+\beta^2)^2 \mathfrak{u}^{(0)} = (\tfrac{1}{2} - 2F)2\beta^2(1+\beta^2) + 4F\chi(\tfrac{2}{7})^{1/2}\beta^2\cos 3\gamma + F[1-(\chi^2/14)]4\beta^4. \tag{29}$$

Potential-Energy Surface.—The potential-energy function displayed in (29) shares most of its salient properties with the polynomial potentials discussed at length in the literature,[22,23] and therefore we can be relatively brief. First of all, independently of β, the minimum occurs for $\sin 3\gamma = 0$, i.e, $\gamma = 0°$ or $60°$. For $F > 0$, we choose $\gamma = 0$. The resulting function is then very similar in its properties to the standard quartic potential, except for the cutoff factors for large β. For sufficiently small F, the minimum of $\mathfrak{u}^{(0)}$ occurs for $\beta = 0$, the spherical, vibrational, or SU(5) limit. As F increases sufficiently to yield a minimum for $\beta \neq 0$, further discussion requires consideration of separate regimes for the parameter χ. In analogy with the standard discussions, $\chi > 0$ corresponds to an oblate deformation, whereas $\chi < 0$ yields a prolate shape. For $\chi = 0$, the potential is γ independent, i.e., we are dealing with the γ-unstable case. The simplest case of deformation to study is the γ-unstable one. The minimum occurs for

$$\beta^2 = (F-4)/(F+4). \tag{30}$$

We consider just one more example, the SU(3) limit corresponding to $\chi = -\frac{1}{2}\sqrt{7}$ and $F \to \infty$. The potential-energy function in this case reaches its minimum for $\beta = \sqrt{2}$. We leave it as an exercise to show that β becomes negative for positive χ, corresponding to an oblate deformation.

The results described above for the potential-energy surface agree with those given in Refs. 5 and 8. This is not to imply that our results continue to agree beyond this approximation, since no continuation method is given in Ref. 5, and Ref. 8 utilizes the method of generator coordinates which yields a non–self-adjoint Hamiltonian. It is a matter of principle that such a Hamiltonian is not the Bohr phenomenological Hamiltonian. In a paper received[24] after this manuscript was submitted for the first time, it is shown that the Hamiltonian in Ref. 8 becomes self-adjoint in the limit $N \to \infty$. This also explains why the potential-energy surfaces agree, since *ours* is manifestly correct. However, our Hamiltonian is self-adjoint for *any* value of N. These differences will be elaborated in a more complete publication.

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Quadrupole Effects in ^{7}Li and ^{9}Be Scattering and the Folding Model

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The inclusion of the quadrupole moment of ^{7}Li and ^{9}Be in coupled-channels calculations removes the need to renormalize the real double-folded potential, obtained from an effective nucleon-nucleon interaction, for ^{7}Li + ^{54}Fe, ^{40}Ca and ^{9}Be + ^{40}Ca elastic scattering.

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In the double-folding model, the optical potential is obtained by folding an effective nucleon-nucleon interaction with the projectile- and target-density distributions. Elastic scattering of the lighter heavy ions has been in general successfully described[1-3] by using the effective nucleon-nucleon interaction M3Y,[4] which is based on a realistic G matrix, to generate a double-folded real part of the optical potential. However, the scattering of ^{6}Li (Refs. 5 and 6) and ^{9}Be (Ref. 7) projectiles appears to be anomalous in the sense that the M3Y interaction has to be reduced by a factor of about 2 to reproduce the data. Recently, the need for a renormalization of the double-folded potential by a factor of $\sim\frac{1}{2}$ has been shown for ^{7}Li scattering.[8,9] Satchler[7] has suggested that the anomalous behavior of ^{6}Li and ^{9}Be could be connected with the very small breakup energies of these two nuclei: only 1.47 MeV and 1.57 MeV for $^6\mathrm{Li} \rightarrow \alpha + d$ and $^9\mathrm{Be} \rightarrow 2\alpha + n$, respectively. The nuclei ^{7}Li and ^{9}Be have, however, another important property, large static quadrupole moments: $-4.5 \pm 0.5\ e\cdot\mathrm{fm}^2$ for ^{7}Li (Ref. 10) and $+4.9 \pm 0.3\ e\cdot\mathrm{fm}^2$ for ^{9}Be (Ref. 11). Blair[12] first suggested that ground-state quadrupole moments could be important in elastic scattering of heavy ions. Recent work[13] has demonstrated that there are significant quadrupole contributions to ^{10}B elastic scattering. In this Letter we show that when the strong quadrupole effects in the scattering of ^{7}Li and ^{9}Be projectiles are treated explicitly in coupled-channels calculations, no renormalization of the real double-folded potential is needed to reproduce the data in the cases investigated.

The targets ^{40}Ca and ^{54}Fe were chosen for the present study to minimize the role of strongly coupled target excited states, which could obscure the effects due to the projectile quadrupole moment. We analyzed previously measured data for ^{7}Li + ^{54}Fe elastic scattering and inelastic scattering to the first excited state in ^{7}Li taken at $E_{\mathrm{lab}} = 48$ MeV (Ref. 14), and for ^{7}Li + ^{40}Ca elastic scattering at 34 MeV (Ref. 15), and our new data for ^{9}Be + ^{40}Ca elastic scattering at 40 MeV. The ^{9}Be + ^{40}Ca data were taken at the Florida State University tandem laboratory and details of these

